FUNCTION SPACES AND ELLIPTIC OPERATORS

(Espacios de funciones y operadores elípticos)

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Contents

INTRODUCTION

The main aim of this thesis is to continue with the theory of Hardy spaces associated with an elliptic operator L. This theory was initiated by P.Auscher, X. T. Duong, and A. McIntosh in [5], in an unpublished work. Besides, P.Auscher and E. Russ in [17] contemplated the possibility of replacing the Laplacian with another second elliptic operator L, in the definition of the Hardy space $H^1(\mathbb{R}^n)$ via the maximal function associated with the Poisson semigroup generated by Δ, the Laplace operator on $\mathbb{R}^n$. This was also considered in dimension one by P. Auscher and P. Tchamitchian in [16]. In [17] the authors showed that, effectively, that generalization could be done in higher dimensions and on domains, provided L satisfies a smooth condition and pointwise Gaussian bounds. In particular, any real elliptic operator satisfy the conditions imposed in [17]. In the context of complete connected Riemannian manifolds with doubling Riemannian measure, P. Auscher, A. McIntosh, and E. Russ in [12] defined Hardy spaces for differential operators and gave different characterizations of them, including atomic decompositions. In that paper they also introduced the use of off-diagonal estimates instead of Gaussian bounds, and the notions of molecules and molecule decomposition (in the context of differential operators) that replace those of atoms and atomic decompositions. The molecules are functions that belong to the range of L^M for M large enough (this condition replaces the cancellation condition in the case of atoms). Another difference with atoms is that molecules do not have compact support, but they decay sufficiently fast. Simultaneously, in the Euclidean setting, S. Hofmann and S. Mayboroda in [55], for $p = 1$, and S. Hofmann, S. Mayboroda, and A. McIntosh in [56], for a general p, and at the same time by an article of R. Jiang and D. Yang, see [61], developed the Hardy space theory for Hardy spaces defined via the Riesz transform associated with a second order divergence form elliptic operator L, and via conical square functions and non-tangential maximal functions associated with the heat and Poisson semigroups generated by the operator L.

Here, we consider weighted Euclidean spaces for Muckenhoupt weights, and give natural extensions, in the weighted context, of the definitions of the Hardy spaces associated with conical square functions, non-tangential maximal functions, and the Riesz transform considered in [55] and [56]. We develop a systematic study of those weighted Hardy spaces which contains the unweighted case as a particular example.

During the study of these weighted Hardy spaces other questions more or less related to this Hardy space theory arise. For example, the study of the boundedness on $L^p(w)$ of the conical square functions and the non-tangential maximal functions associated with the operator L (essential for the development of the Hardy space theory); or the study of tent spaces and weighted tent spaces. We study these spaces, which appear naturally in the context of this work, and also show how certain operators behave when they are applied to functions that belong to them.

In what follows we introduce the main results presented in this work. For historical background and a summary of main results on the field, we refer the reader to the comments we make at the beginning of each section.

The weights that we consider are positive locally integrable functions w belonging to the Muckenhoupt classes A_r or the reverse Hölder classes RH_q (see Section 1.1 for definitions). Besides we denote by L an elliptic operator in divergence form, defined by

$$Lf := -\operatorname{div}(A\nabla f),$$

where A is an $n\times n$ matrix with complex bounded coefficients, satisfying an ellipticity condition. The operator $-L$ generates a C^0-semigroup of contractions on $L^2(\mathbb{R}^n)$: $\{e^{-tL}\}_{t>0}$, called the heat semigroup. Associated with this semigroup and its gradient we consider the following conical square functions:

$$\mathcal{S}_{m,\mathrm{H}}f(x) = \left(\iint_{\Gamma(x)} |(t^2L)^m e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

and, for every $m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$,

$$\mathrm{G}_{m,\mathrm{H}}f(x) = \left(\iint_{\Gamma(x)} |t\nabla_y (t^2L)^m e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$\mathcal{G}_{m,\mathrm{H}}f(x) = \left(\iint_{\Gamma(x)} |t\nabla_{y,t} (t^2L)^m e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}};$$

and also the ones associated with the Poisson semigroup:

$$\mathcal{S}_{K,\mathrm{P}}f(x) = \left(\iint_{\Gamma(x)} |(t\sqrt{L})^{2K} e^{-t\sqrt{L}} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

and for every $K \in \mathbb{N}_0$,

$$\mathrm{G}_{K,\mathrm{P}}f(x) = \left(\iint_{\Gamma(x)} |t\nabla_y (t\sqrt{L})^{2K} e^{-t\sqrt{L}} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$\mathcal{G}_{K,\mathrm{P}}f(x) = \left(\iint_{\Gamma(x)} |t\nabla_{y,t} (t\sqrt{L})^{2K} e^{-t\sqrt{L}} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma(x) := \{(y,t)\in\mathbb{R}^{n+1}_+ : |x-y| < t\}$ is the cone of aperture one with vertex at x. Corresponding to the cases $m=0$ or $K=0$ we simply write $\mathrm{G}_{\mathrm{H}}f := \mathrm{G}_{0,\mathrm{H}}f$, $\mathcal{G}_{\mathrm{H}}f := \mathcal{G}_{0,\mathrm{H}}f$, $\mathrm{G}_{\mathrm{P}}f := \mathrm{G}_{0,\mathrm{P}}f$, and $\mathcal{G}_{\mathrm{P}}f := \mathcal{G}_{0,\mathrm{P}}f$. Besides, we set $\mathcal{S}_{\mathrm{H}}f := \mathcal{S}_{1,\mathrm{H}}f$, $\mathcal{S}_{\mathrm{P}}f := \mathcal{S}_{1,\mathrm{P}}f$.

We also consider the following non-tangential maximal functions:

$$\mathcal{N}_{\mathrm{H}}f(x) = \sup_{(y,t)\in\Gamma(x)} \left(\int_{B(y,t)} |e^{-t^2L}f(z)|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}} \quad\text{and}\quad \mathcal{N}_{\mathrm{P}}f(x) = \sup_{(y,t)\in\Gamma(x)} \left(\int_{B(y,t)} |e^{-t\sqrt{L}}f(z)|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}}.$$

Finally, we denote the Riesz transform associated with the operator L by $\nabla L^{-\frac{1}{2}}$. This operator has the integral representation:

$$\nabla L^{-\frac{1}{2}} h = \frac{1}{\pi^{1/2}} \int_0^\infty \sqrt{t}\nabla e^{-tL} h \frac{dt}{t}.$$

For $\mathcal{T}$ denoting any of the previous operators, the Hardy space associated with $\mathcal{T}$, $H^p_{\mathcal{T},q}(w)$, is defined as the completion of the set

$$\mathbb{H}^p_{\mathcal{T},q}(w) := \{f \in L^q(w) : \mathcal{T}f \in L^p(w)\},$$

with respect to the quasi-norm $\|f\|_{\mathbb{H}^p_{\mathcal{T},q}(w)} := \|\mathcal{T}f\|_{L^p(w)}$. We take $0<p<\infty$ and, for $\mathcal{T}$ being a square function or a non-tangential maximal function, we take $q \in \mathcal{W}_w(p_-(L),p_+(L))$, in the case that $\mathcal{T}$ is equal to the Riesz transform we take $q \in \mathcal{W}_w(q_-(L),q_+(L))$. We have that $(p_-(L),p_+(L))$ and $(q_-(L),q_+(L))$ are the intervals where the heat semigroup generated by L, and its gradient are respectively uniformly bounded on $L^p(\mathbb{R}^n)$. In general, for some $0 < p_0 < q_0 < \infty$, $\mathcal{W}_w(p_0, q_0)$ is the possibly empty interval of $q \in (p_0, q_0)$ such that $w \in A_{\frac{q}{p_0}} \cap RH_{\left(\frac{q_0}{q}\right)'}$. Throughout this text, when we take a point in $\mathcal{W}_w(p_0, q_0)$, we implicitly understand that we are working with a weight $w \in A_\infty$ such that $\mathcal{W}_w(p_0, q_0) \neq \emptyset$, and therefore we can take that point.

In Chapter 4, we show that for $0 < p \leq 1$ and $\mathcal{T}$ being any conical square function or a non-tangential maximal function, we have a molecular characterization of the weighted Hardy spaces $H^p_{\mathcal{T},q}(w)$. By this, we mean that in that range of p's these spaces are isomorphic to a molecular weighted Hardy space $H^p_L(w)$ defined as follows: let $w \in A_\infty$, $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p < \infty$, $\varepsilon > 0$, and $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$ ($r_w := \inf\{1 \leq r < \infty : w \in A_r\}$), we define the molecular weighted Hardy space $H^p_{L,q,\varepsilon,M}(w)$ as the completion of the set

$$\mathbb{H}^p_{L,q,\varepsilon,M}(w) := \left\{ \sum_{i=1}^{\infty} \lambda_i \mathfrak{m}_i : \sum_{i=1}^{\infty} \lambda_i \mathfrak{m}_i \text{ is a } (w, q, p, \varepsilon, M)\text{-representation} \right\},$$

with respect to the quasi-norm,

$$\|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)} := \inf \left\{ \left(\sum_{i=1}^{\infty} |\lambda_i|^p \right)^{\frac{1}{p}} : \sum_{i=1}^{\infty} \lambda_i \mathfrak{m}_i \text{ is a } (w, q, p, \varepsilon, M)\text{-representation of } f \right\}.$$

Where, we say that a function $\mathfrak{m} \in L^q(w)$ (belonging to the range of L^k in $L^q(w)$, $0 \leq k \leq M$,) is a $(w, q, p, \varepsilon, M)$-molecule if, for some cube $Q \subset \mathbb{R}^n$, $\mathfrak{m}$ satisfies

$$\|\mathfrak{m}\|_{mol,w} := \sum_{i \geq 1} 2^{i\varepsilon} w(2^{i+1}Q)^{\frac{1}{p} - \frac{1}{q}} \sum_{k=0}^{M} \left\| \left((\ell(Q)^2 L)^{-k} \mathfrak{m} \right) \mathbf{1}_{C_i(Q)} \right\|_{L^q(w)} < 1.$$

Henceforth, we refer to the previous expression as the molecular w-norm of $\mathfrak{m}$ and any cube Q satisfying that, is called a cube associated with $\mathfrak{m}$.

Note that if $\mathfrak{m}$ is a $(w, q, p, \varepsilon, M)$-molecule, for all associated cubes Q:

$$\left\| \left((\ell(Q)^2 L)^{-k} \mathfrak{m} \right) \mathbf{1}_{C_i(Q)} \right\|_{L^q(w)} \leq 2^{-i\varepsilon} w(2^{i+1}Q)^{\frac{1}{q} - \frac{1}{p}} \quad i = 1, 2, \ldots; \ k = 0, 1, \ldots, M;$$

(see (4.6) for the definition of $C_j(Q_i)$). Besides, for any function f, we say that the sum $\sum_{i \in \mathbb{N}} \lambda_i \mathfrak{m}_i$ is a $(w, q, p, \varepsilon, M)$-representation of f, if the following conditions are satisfied:

(i) $\{\lambda_i\}_{i \in \mathbb{N}} \in \ell^p$.

(ii) For every $i \in \mathbb{N}$, $\mathfrak{m}_i$ is a $(w, q, p, \varepsilon, M)$-molecule.

(iii) $f = \sum_{i \in \mathbb{N}} \lambda_i \mathfrak{m}_i$ in $L^q(w)$.

In Remark 4.12 we fix a choice of the parameters p, q, ε, and M and denote the corresponding molecular weighted Hardy space by $H^p_L(w)$. The molecular characterization that we announced above is stated in the next results.

Theorem 1. *Given $w \in A_\infty$ and $0 < p \leq 1$, let $H^p_L(w)$ be the fixed molecular Hardy space as in Remark 4.12. For every $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $\varepsilon > 0$, and $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$, the following spaces are isomorphic to $H^p_L(w)$ (and therefore one another) with equivalent norms*

$$H^p_{L,q,\varepsilon,M}(w); \qquad H^p_{\mathcal{S}_{m,\mathrm{H}},q}(w),\ m \in \mathbb{N}; \qquad H^p_{\mathrm{G}_{m,\mathrm{H}},q}(w),\ m \in \mathbb{N}_0; \quad \text{and} \qquad H^p_{\mathcal{G}_{m,\mathrm{H}},q}(w),\ m \in \mathbb{N}_0.$$

In particular, none of these spaces depend (modulo isomorphisms) on the choice of the allowable parameters q, ε, M, and m.

Theorem 2. *Given $w \in A_\infty$ and $0 < p \leq 1$, let $H^p_L(w)$ be the fixed molecular Hardy space as in Remark 4.12. For every $q \in \mathcal{W}_w(p_-(L), p_+(L))$, the following spaces are isomorphic to $H^p_L(w)$ (and therefore one another) with equivalent norms*

$$H^p_{\mathcal{S}_{K,\mathrm{P}},q}(w),\ K \in \mathbb{N}; \qquad H^p_{\mathrm{G}_{K,\mathrm{P}},q}(w),\ K \in \mathbb{N}_0; \qquad \text{and} \qquad H^p_{\mathcal{G}_{K,\mathrm{P}},q}(w),\ K \in \mathbb{N}_0.$$

In particular, none of these spaces depend (modulo isomorphisms) on the choice of q, and K.

Theorem 3. *Given $w \in A_\infty$ and $0 < p \leq 1$, let $H^p_L(w)$ be the fixed molecular Hardy space as in Remark 4.12. For every $q \in \mathcal{W}_w(p_-(L), p_+(L))$, the following spaces are isomorphic to $H^p_L(w)$ (and therefore one another) with equivalent norms*

$$H^p_{\mathcal{N}_{\mathrm{H}},q}(w) \qquad \text{and} \qquad H^p_{\mathcal{N}_{\mathrm{P}},q}(w).$$

In particular, none of these spaces depend (modulo isomorphisms) on the choice of q.

When $p \in \mathcal{W}_w(p_-(L), p_+(L))$, we find out that these spaces are isomorphic with the $L^p(w)$ spaces.

Theorem 4. *Given $w \in A_\infty$, if $\mathcal{T}$ is any of the square functions in* (1.26)-(1.31) *or a non-tangential maximal function in* (1.32)*, we have, for all $p \in \mathcal{W}_w(p_-(L), p_+(L))$, that the spaces $H^p_{\mathcal{T}}(w)$ and $L^p(w)$ are isomorphic, with equivalent norms.*

In the case of the Riesz transform we obtain, for some range of p's, an isomorphism between $H^p_{\nabla L^{-1/2},q}(w)$ and $H^p_{\mathcal{S}_{\mathrm{H}},q}(w)$.

Theorem 5. *Given $w \in A_\infty$ such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. For all* $\max\left\{r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)}\right\} < p < \frac{q_+(L)}{s_w}$ *and $q \in \mathcal{W}_w(q_-(L), q_+(L))$, we have the following isomorphism*

$$H^p_{\mathcal{S}_{\mathrm{H}},q}(w) \approx H^p_{\nabla L^{-1/2},q}(w).$$

For $p \in \mathcal{W}_w(q_-(L), q_+(L))$ this isomorphism does not depend on q.

In order to prove these theorems we need to develop an $L^p(w)$ theory for the operators which we are working with. That is, we need to know in which range of $p' s$ these operators are bounded on $L^p(w)$. In [11] the authors studied this boundedness in the case of the Riesz transform. In fact, that paper was the third of a series of four papers of which the first two: [9] and [10], are also of our interest. In them they developed a weighted theory related with different objects that arise connected with elliptic operators in divergence form. These are the heat semigroup, its gradient, functional calculus, the Riesz transform, and the vertical square functions defined by the mentioned semigroups. They also gave very useful tools, and, in fact, crucial to the present work when dealing with those objects, such as: off-diagonal estimates and extrapolation. Moreover, they introduced the notation that we shall mainly use in this manuscript.

We should remark that we shall not only use weighted theory, also the unweigthed off-diagonal estimates proved by P. Auscher in [3] and the boundedness of the vertical and conical square functions (see [3] and [7]) will be very useful in our work.

Coming back to our problem, recall that we needed an $L^p(w)$ theory for the conical square functions and non-tangential maximal functions. We solve this problem in Chapter 3, where we show weighted boundedness for the conical square function and the non-tangential maximal function that we have defined, as well as norm comparison.

Theorem 6. *Let $w \in A_\infty$.*

(*a*) *$\mathcal{S}_{\mathrm{H}}$, G_{H}, and $\mathcal{G}_{\mathrm{H}}$ are bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), \infty)$.*

(*b*) *Given $m \in \mathbb{N}$, $\mathcal{S}_{m,\mathrm{H}}$, $\mathrm{G}_{m,\mathrm{H}}$, and $\mathcal{G}_{m,\mathrm{H}}$ are bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), \infty)$.*

Equivalently, all the previous square functions are bounded on $L^p(w)$ *for every* $p_-(L) < p < \infty$ *and every* $w \in A_{\frac{p}{p_-(L)}}$.

Theorem 7. *Let* $w \in A_\infty$.

(*a*) *Given* $K \in \mathbb{N}$, $\mathcal{S}_{K,\mathrm{P}}$ *is bounded on* $L^p(w)$ *for all* $p \in \mathcal{W}_w(p_-(L), p_+(L)^{K,*})$.

(*b*) *Given* $K \in \mathbb{N}_0$, $\mathcal{G}_{K,\mathrm{P}}$ *and* $\mathrm{G}_{K,\mathrm{P}}$ *are bounded on* $L^p(w)$ *for all* $p \in \mathcal{W}_w(p_-(L), p_+(L)^{K,*})$.

Theorem 8. *Given* $w \in A_\infty$. *There hold*

(*a*) $\mathcal{N}_\mathrm{H}$ *is bounded on* $L^p(w)$ *for all* $p \in \mathcal{W}_w(p_-(L), \infty)$,

(*b*) $\mathcal{N}_\mathrm{P}$ *is bounded on* $L^p(w)$ *for all* $p \in \mathcal{W}_w(p_-(L), p_+(L))$.

The comparison results are useful in both the characterization of our weighted Hardy spaces and in showing boundedness on $L^p(w)$ of the operators in the left-hand side of the inequality.

Theorem 9. *Given an arbitrary* $f \in L^2(\mathbb{R}^n)$ *there hold:*

(*a*) $\mathrm{G}_{m,\mathrm{H}} f(x) \le \mathcal{G}_{m,\mathrm{H}} f(x)$, *for every* $x \in \mathbb{R}^n$ *and for all* $m \in \mathbb{N}_0$.

(*b*) *Given* $m \in \mathbb{N}$, $\|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_\mathrm{H} f\|_{L^p(w)}$, *for all* $w \in A_\infty$ *and* $0 < p < \infty$.

(*c*) *Given* $m \in \mathbb{N}$, $\|\mathcal{G}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_\mathrm{H} f\|_{L^p(w)}$, *for all* $w \in A_\infty$ *and* $0 < p < \infty$.

Theorem 10. *Given an arbitrary* $f \in L^2(\mathbb{R}^n)$ *there hold:*

(*a*) $\mathrm{G}_{K,\mathrm{P}} f(x) \le \mathcal{G}_{K,\mathrm{P}} f(x)$, *for every* $x \in \mathbb{R}^n$ *and for all* $K \in \mathbb{N}_0$.

(*b*) *Given* $K \in \mathbb{N}$, $\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_\mathrm{H} f\|_{L^p(w)}$, *for all* $w \in A_\infty$ *and* $p \in \mathcal{W}_w(0, p_+(L)^{K,*})$.

(*c*) $\|\mathcal{G}_\mathrm{P} f\|_{L^p(w)} \lesssim \|\mathcal{G}_\mathrm{H} f\|_{L^p(w)}$, *for all* $w \in A_\infty$ *and* $w \in \mathcal{W}_w(0, p_+(L)^*)$.

(*d*) *Given* $K \in \mathbb{N}$, $\|\mathcal{G}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_\mathrm{H} f\|_{L^p(w)}$, *for all* $w \in A_\infty$ *and* $p \in \mathcal{W}_w(0, p_+(L)^{K,*})$.

Theorem 11. *For all* $w \in A_\infty$, $q \in \mathcal{W}_w(p_-(L), p_+(L))$, *and* $f \in L^q(w)$. *There hold*

(*a*) $\|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathrm{G}_{m-1,\mathrm{H}} f\|_{L^p(w)}$, *for all* $m \in \mathbb{N}$ *and* $0 < p < \infty$,

(*b*) $\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^{p_0}(w)} \lesssim \|\mathrm{G}_{K-1,\mathrm{P}} f\|_{L^{p_0}(w)}$, *for all* $K \in \mathbb{N}$ *and* $0 < p < \infty$.

Theorem 12. *Given an arbitrary* $f \in L^2(\mathbb{R}^n)$, *there hold, for all* $w \in A_\infty$ *and* $0 < p < \infty$,

(*a*) $\|\mathcal{G}_\mathrm{P} f\|_{L^p(w)} \lesssim \|\mathcal{N}_\mathrm{P} f\|_{L^p(w)}$,

(*b*) $\|\mathcal{S}_\mathrm{H} f\|_{L^p(w)} \lesssim \|\mathcal{N}_\mathrm{H} f\|_{L^p(w)}$.

The proofs of all these results rely on the theory of tent spaces. In Chapter 2 we briefly recall that theory, introduced in [32] by R. R. Coifman, Y. Meyer, and E. M. Stein.

For $0 < r, p < \infty$, the tent space T_r^p is defined by

$$T_r^p := \{F \text{ measurable functions in } \mathbb{R}^{n+1}_+ : \mathcal{A}_r F \in L^p(\mathbb{R}^n)\}$$

where

$$\mathcal{A}_r F(x) := \left(\iint_{\Gamma(x)} |F(y,t)|^r \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{r}}.$$

These spaces are quasi-Banach spaces endowed with the norm $\|F\|_{T_r^p} := \|\mathcal{A}F\|_{L^p(\mathbb{R}^n)}$, the functions in $L^r(\mathbb{R}_+^{n+1})$ with compact support are dense in them, and their definitions do not depend on the aperture of the cone $\Gamma^\alpha(x) := \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < \alpha t\}$, $\alpha > 0$, used to define the operator $\mathcal{A}_r F$ (see [32, 4]).

Besides, we also define the weak tent spaces wT_r^p as the usual tent spaces but imposing that the operator $\mathcal{A}_r F \in L^{p,\infty}(\mathbb{R}^n)$. Finally, for $0 < p \leq 1$, we define the space $\mathfrak{T}_r^p$ as the subspace of functions $F \in T_r^p$ having an atomic decomposition $\sum_{i=1}^\infty \lambda_i A_i$ such that the atoms A_i also satisfy $\int_{\mathbb{R}^n} A_i(x,t)dx = 0$ for a.e. $t > 0$ and $\forall i \geq 1$, and that $\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{\frac{1}{p}} < \infty$.

We study how certain well known operators such as the Hardy-Littlewood maximal operator, Calderón-Zydmund operators, Riesz potentials, fractional maximal functions, and the Riesz transform, act on those spaces.

Theorem 13. *Let $\mathcal{M}$ be the centered Hardy-Littlewood maximal operator. For all $1 < r < \infty$,*

(*a*) $\mathcal{M} : T_r^p \to T_r^p$, *for all* $1 < p < \infty$;

(*b*) $\mathcal{M} : T_r^1 \to wT_r^1$.

Theorem 14. *Let $\mathcal{T}$ be a Calderón-Zygmund operator on $\mathbb{R}^n$ of order $\delta \in (0,1]$. For all $1 < r < \infty$,*

(*a*) $\mathcal{T} : T_r^p \to T_r^p$, *for all* $1 < p < \infty$;

(*b*) $\mathcal{T} : T_r^1 \to wT_r^1$;

(*c*) $\mathcal{T} : \mathfrak{T}_r^p \to T_r^p$, *for all* $\frac{n}{n+\delta} < p \leq 1$;

(*d*) $\mathcal{T} : \mathfrak{T}_r^p \to \mathfrak{T}_r^p$, *for all* $\frac{n}{n+\delta} < p \leq 1$, *if* $\mathcal{T}^*(1) = 0$.

Theorem 15. *For $0 < \alpha < n$, $\frac{n}{n-\alpha} < r < \infty$, and $1 < p < q < \infty$ such that $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$,*

$$\mathcal{I}_\alpha, \mathcal{M}_\alpha : T_r^p \to T_r^q.$$

Theorem 16. *Let $L = -\operatorname{div}(A\nabla)$ be an elliptic operator with complex-valued coefficients. For $q_-(L) < p, r < q_+(L)$ we have*

$$\nabla L^{-\frac{1}{2}} : T_r^p \to T_r^p.$$

In the last section of Chapter 2, we consider weighted tent spaces as follows: for $0 < r, p < \infty$

$$T_r^p(w) := \{F \text{ measurable functions in } \mathbb{R}_+^{n+1} : \mathcal{A}_r F \in L^p(w)\}.$$

These spaces are also quasi-Banach spaces when endowed with the norm $\|F\|_{T_r^p(w)} := \|\mathcal{A}F\|_{L^p(w)}$, the functions in $L^r(\mathbb{R}_+^{n+1})$ with compact support are also dense on them, and their definitions, again, do not depend on the aperture of the cone $\Gamma^\alpha(x)$, used to define the operator $\mathcal{A}_r F$.

Proposition 17. *Let $0 < \alpha \leq \beta < \infty$.*

(*i*) *For every $w \in A_{\widehat{r}}$, $1 \leq \widehat{r} < \infty$, there holds*

$$\|\mathcal{A}^\beta F\|_{L^p(w)} \leq C \left(\frac{\beta}{\alpha}\right)^{\frac{n\widehat{r}}{p}} \|\mathcal{A}^\alpha F\|_{L^p(w)} \quad \textit{for all} \quad 0 < p \leq 2\widehat{r}.$$

(*ii*) *For every $w \in RH_{s'}$, $1 \leq s < \infty$, there holds*

$$\|\mathcal{A}^\alpha F\|_{L^p(w)} \leq C \left(\frac{\alpha}{\beta}\right)^{\frac{n}{sp}} \|\mathcal{A}^\beta F\|_{L^p(w)} \quad \textit{for all} \quad \frac{2}{s} \leq p < \infty.$$

The main purpose of the last section in Chapter 2 is to prove a complex interpolation result for weighted tent spaces:

Theorem 18. *Suppose* $1 \le p_0 < p_1 < \infty$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$. *Then*

$$[T^{p_0}(w), T^{p_1}(w)]_\theta = T^p(w).$$

We use this theorem to obtain complex interpolation between weighted Hardy spaces:

Theorem 19. *Given* $w \in A_\infty$ *and* $q \in \mathcal{W}_w(p_-(L), p_+(L))$. *Suppose* $1 \le p_0 < p_1 < \frac{p_+(L)^{1/2,*}}{s_w}$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$. *Then*

$$[H^{p_0}_{\mathcal{S}_{\rm H},q}(w), H^{p_1}_{\mathcal{S}_{\rm H},q}(w)]_\theta = H^{p}_{\mathcal{S}_{\rm H},q}(w).$$

This result will be needed in the proof of Theorem 5.

We finally comment that Chapter 1 is devoted to introduce all the definitions and concepts needed to understand the next chapters, such as: weights, elliptic operators in diverge form, off-diagonal estimates, and extrapolation. Therefore, if the reader is familiar with these concepts they can just skip that chapter.

This thesis has led to the following papers:

1. P. Auscher, C. Prisuelos-Arribas, *Tent space boundedness via extrapolation*, Mathematische Zeitschrift (2016), DOI 10.1007/s00209-016-1814-7.

2. J.M. Martell, C. Prisuelos-Arribas, *Weighted Hardy spaces associated with elliptic operators. Part I: weighted norm inequalities for conical square functions*, to appear in Trans. Amer. Soc. arXiv:1406.6285.

3. J.M. Martell, C. Prisuelos-Arribas, *Weighted Hardy spaces associated with elliptic operators. Part II: Characterizations of* $H^1_L(w)$, to appear in Publ. Mat., arXiv:1701.00920.

4. C. Prisuelos-Arribas *Weighted Hardy spaces associated with elliptic operators. Part III: Characterizations of* $H^p_L(w)$ *and the weighted Hardy space associated with the Riesz transform*. In preparation.

In the first one, we explore the boundedness of different operator on tent spaces. This is develop in Chapter 2, Section 2.2; and Appendix A. The main results are Theorems 13, 14, 15, and 16.

In the second one, we start elaborating a theory of weighted Hardy spaces associated with the operator L by proving boundedness on $L^p(w)$ of the conical square functions defined above. We also obtain some norm comparison results between those square functions, and change of angles in weighted tent spaces. The main results are Theorems 6, 7, 9, and 10, and Proposition 17. This appears in Chapter 2: Sections 2.3.1 and 2.3.2; and in Chapter 3, Section 3.1.

In the third one, we develop an $L^p(w)$ theory for the non-tangential maximal functions defined above. Besides, for $p = 1$, we define the weighted Hardy spaces associated with conical square functions and non-tangential maximal functions and give a molecular characterization of them. This is in Chapter 3, Section 3.1 and 3.2; and in Chapter 4, Sections 4.1 and 4.3 with $p = 1$. We highlight Theorems 1, 2, 3, 8, 11, and 12.

In the last one, we continue with the theory of weighted Hardy spaces for p different from one, and additionally we consider the weighted Hardy space defined using the Riesz transform associated with the operator L. The results are contained in Chapter 2, Section 2.3.3; Chapter 3, Section 3.3, and Chapter 4 (recalling that the case $p = 1$ in Sections 4.1 and 4.3 appears in the third paper). We bring to focus Theorems 1, 2, and 3 with $0 < p < 1$, and Theorems 4, 5, 18, and 19.

NOTATION

Although the objects listed here are also defined in the text, we include this section to facilitate a quick search.

(*a*) We write C or c to indicate that the constants are independent of the decisive parameters in the corresponding computation. Sometimes, we also write c_w or $c_{n,w}$, this means that the constant may depends on the weight, or on the weight and the dimension, but that dependence is not relevant in the corresponding computation.

(*b*) Given, two quantities Θ_1 and Θ_2, we write $\Theta_1 \lesssim \Theta_2$ or $\Theta_1 \approx \Theta_2$ to symbolize $\Theta_1 \leq C\Theta_2$ or $\Theta_1 = C\Theta_2$.

(*c*) We represent by $\mathbb{N}_0$ the set of natural numbers including zero.

(*d*) For any $1 \leq r < \infty$, we denote by r' its conjugated exponent: $1/r + 1/r' = 1$.

(*e*) The upper half space is represent by $\mathbb{R}^{n+1}_+ := \{(y,t) : \, y \in \mathbb{R}^n \; t > 0\}$.

(*f*) Given $0 \leq p_0 < q_0 \leq \infty$ we can consider the interval $\mathcal{W}_w(p_0, q_0) := \{p \in (p_0, q_0) : w \in A_{p/p_0} \cap RH_{(q_0/p)'}\}$.

(*g*) For $\alpha > 0$, the cone of aperture α is defined as $\Gamma(x)^\alpha := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. In the case that $\alpha = 1$ we write $\Gamma(x)$.

(*h*) Let F be a function defined in $\mathbb{R}^{n+1}_+$ we denote the integral of F over a cone by

$$|||F|||_{\Gamma(x)} := \left(\iint_{\Gamma(x)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

(*i*) For a close set $E \subset \mathbb{R}^n$ and $\alpha > 0$, the union of all cones with vertex at E is denote by $\mathcal{R}^\alpha(E) := \cup_{x \in E} \Gamma^\alpha(x)$. We write $\mathcal{R}(E)$ when $\alpha = 1$.

(*j*) For an open set $O \subset \mathbb{R}^n$ we defined the tent over O as the set $\widehat{O} := \{(y,t) \in \mathbb{R}^{n+1}_+ : d(y, \mathbb{R}^n \setminus O) \geq t\}$,

(*k*) We write $Q = Q(x_Q, \ell(Q))$ to denote the cube of centre x_Q and sidelength $\ell(Q)$.

(*l*) We write $B = B(x_B, r_B)$ to denote the ball with centre x_B and radius r_B.

(*m*) $CQ = Q(x_Q, C\ell(Q))$ denotes the cube of centre x_Q and sidelength C times $\ell(Q)$.

(*n*) $CB = B(x_B, Cr_B)$ denotes the ball with centre x_B and radius C times r_B.

(*o*) Let E be a ball or a cube, we define the annuli of E by $C_j(E) = 2^{j+1}E \setminus 2^jE$, $j \geq 2$, $C_1(E) = 4E$, .

(*p*) For any measurable set $E \subset \mathbb{R}^n$ and any weight $w \in A_\infty$ we set

$$w(E) := \int_E w(x)dx.$$

(*q*) The space $L^p(w)$ denotes the space of functions f such that $fw^{1/p} \in L^p(\mathbb{R}^n)$.

(*r*) We represent the average of a function h in a set E under the measure μ by $\fint_E h(x)d\mu = \frac{1}{\mu(E)} \int_E h(x)d\mu$. In this work μ shall be the Lebesgue measure or a weight $w \in A_\infty$. When μ is the Lebesgue measure or a weight $w \in A_\infty$ we write respectively dx or dw instead of $d\mu$.

(*s*) When we have an integral over an annulus $C_j(E)$ (where E is a ball or a cube in $\mathbb{R}^n$), we denote by $\fint_{C_j(E)} h(x)d\mu = \frac{1}{\mu(2^{j+1}E)} \int_E h(x)d\mu$, the integral over the annulus $C_j(E)$ divided by the measure of the whole set $2^{j+1}E$, instead of the measure of the annulus.

(*t*) $\mathcal{L}(X)$ represents the space of linear continuous maps on a Banach space X.

(*u*) We write ∇_y or ∇ to denote the gradient in the n-dimensional variable, $y \in \mathbb{R}^n$, and $\nabla_{y,t}$ to denote the gradient over the variables $(y,t) \in \mathbb{R}^{n+1}_+$.

(*v*) The vertical square function $\mathrm{g_H}$ is defined by $\mathrm{g_H} f(x) := \left(\int_0^\infty |t^2 L e^{-t^2 L} f(x)|^2\right)^{\frac{1}{2}}$.

(*w*) $\mathcal{M}$ and $\mathcal{M}_u$ denote respectively the centred and uncentred Hardy-Littlewood maximal operators over balls.

(*x*) $\mathcal{M}_d$ and $\mathcal{M}_c$ denote respectively the dyadic maximal operator and the centred maximal operator over cubes.

(*y*) $\mathcal{M}^w$ denotes the weighted maximal operator over cubes, defined by

$$\mathcal{M}^w f(x) := \sup_{Q \ni x} \fint_Q |f(y)|dw.$$

Chapter 1
PRELIMINARIES

1.1 Muckenhoupt weights

A Muckenhoupt weight w is a locally integrable positive function, $w : \mathbb{R}^n \to [0, \infty]$ so that the set of points where w takes the values zero and infinity has Lebesgue measure zero. In 1972 in [70], B. Muckenhoupt proved that the Hardy-Littlewood maximal operator defined on cubes $Q \subset \mathbb{R}^n$:

$$\mathcal{M}_c f(x) := \sup_{Q \ni x} \fint_Q |f(y)| dy$$

is bounded on $L^p(w)$ for $1 < p < \infty$, if and only if w satisfies:

$$\left(\fint_Q w(x)\, dx \right) \left(\fint_Q w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} \leq C. \tag{1.1}$$

In short, for a weight w satisfying (1.1) we write $w \in A_p$, and say that w belongs to the A_p class.

Besides, in [58] R. Hunt, B. Muckenhoupt, and R. Wheeden proved in dimension one, that the fact that $w \in A_p$ is also a necessary and sufficient condition for the Hilbert transform to be bounded on $L^p(w)$. In [30], R.R. Coifman and C. Fefferman extended these results to more general singular integrals in higher dimensions. In the same paper, they also proved that for every weight $w \in A_p$, for $1 < p < \infty$, the following reverse Hölder inequality holds

$$\left(\fint_Q w(x)^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \leq C \fint_Q w(x) dx,$$

for all cubes $Q \subset \mathbb{R}^n$ and for positive constants C and $\delta > 0$ independent of Q.

These results are also true using balls instead of cubes. In fact, the definitions over balls of the Hardy-Littlewood maximal function, and the A_p and reverse Hölder conditions are equivalent (up to dimensional constants) to those made over cubes. In this work, it will be more convenient for us to use balls instead of cubes. Specifically, for each $1 < r < \infty$, and r' such that $1/r + 1/r' = 1$, we say that $w \in A_r$ if

$$\left(\fint_B w(x)\, dx \right) \left(\fint_B w(x)^{1-r'}\, dx \right)^{r-1} \leq C, \quad \forall B \subset \mathbb{R}^n. \tag{1.2}$$

On the other hand, for each $1 < s < \infty$, we say that w belongs to the reverse Hölder class s, and write $w \in RH_s$, if

$$\left(\fint_B w(x)^s\, dx \right)^{\frac{1}{s}} \leq C \fint_B w(x)\, dx, \quad \forall B \subset \mathbb{R}^n. \tag{1.3}$$

Besides, if $w \in A_r$, $1 < r < \infty$, for every ball B and every measurable set $E \subset B$,

$$\frac{w(E)}{w(B)} \geq C \left(\frac{|E|}{|B|} \right)^r. \tag{1.4}$$

Note that if we take $r = 1$ in (1.4) we obtain

$$\frac{w(E)}{|E|} \geq C \frac{w(B)}{|B|}. \tag{1.5}$$

From this inequality we can deduce that, for every ball $B \subset \mathbb{R}^n$,

$$\fint_B w(x)\, dx \leq C w(y), \quad \text{for a.e. } y \in B. \tag{1.6}$$

We say that a weight satisfying this is in the A_1 class. Equivalently, for $w \in A_1$ we have that $\mathcal{M}_u w \leq C\, w$, a.e., where $\mathcal{M}_u$ denotes the uncentred Hardy-Littlewood maximal operator over balls in $\mathbb{R}^n$.

On the other hand, if $w \in RH_s$ for $1 < s < \infty$, and $1/s + 1/s' = 1$, note that

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^{\frac{1}{s'}}. \tag{1.7}$$

For $s = \infty$ this condition looks

$$\frac{w(E)}{|E|} \leq C \frac{w(B)}{|B|}, \tag{1.8}$$

from which we can deduce that there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$w(y) \leq C \fint_B w(x)\, dx, \quad \text{for a.e. } y \in B. \tag{1.9}$$

For a weight satisfying this condition we write $w \in RH_\infty$. Notice that we have excluded the case $s = 1$ since the class RH_1 consists of all the weights, and that is the way RH_1 is understood in what follows.

Moreover, consider for all $1 < r < \infty$,

$$[w]_{A_r} := \sup_{B \subset \mathbb{R}^n} \left(\fint_B w(x)\, dx \right) \left(\fint_B w(x)^{1-r'}\, dx \right)^{r-1},$$

and for $r = 1$,

$$[w]_{A_1} := \sup_{B \subset \mathbb{R}^n} \left(\fint_B w(x)\, dx \right) \|w^{-1}\|_{L^\infty(B)}.$$

Then, we have, for all $w \in A_r$ and $1 \leq r < \infty$,

$$\frac{w(E)}{w(B)} \geq [w]_{A_r}^{-1} \left(\frac{|E|}{|B|} \right)^r. \tag{1.10}$$

This implies in particular that w is a doubling measure:

$$w(\lambda B) \leq [w]_{A_r} \lambda^{n\,r} w(B), \qquad \forall\, B,\ \forall\, \lambda > 1. \tag{1.11}$$

In addition, consider for all $1 < s < \infty$,

$$[w]_{RH_s} := \sup_{B \subset \mathbb{R}^n} \left(\fint_B w(x)\, dx \right)^{-1} \left(\fint_B w(x)^s\, dx \right)^{\frac{1}{s}},$$

and for $s = \infty$,

$$[w]_{RH_\infty} := \sup_{B\subset\mathbb{R}^n} \left(\fint_B w(x)\, dx \right)^{-1} \|w\|_{L^\infty(B)}.$$

Then, we have, for all $w \in RH_s$ and $1 < s \leq \infty$,

$$\frac{w(E)}{w(B)} \leq [w]_{RH_s} \left(\frac{|E|}{|B|} \right)^{\frac{1}{s'}}. \tag{1.12}$$

Moreover, note that the A_r classes increase as r increases, therefore it is natural to consider the limit as r tends to infinity. This is the A_∞ class. We say that a weight is in the A_∞ class if there exists $C > 0$ such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\fint_B w(x)dx \right) \exp \left(\fint_B \log w(x)^{-1} dx \right) \leq C,$$

and denote

$$[w]_{A_\infty} := \sup_{B\subset\mathbb{R}^n} \left(\fint_B w(x)dx \right) \exp \left(\fint_B \log w(x)^{-1} dx \right).$$

It turns out that the A_p, A_∞, and RH_s classes are closely related. We sum up some of the properties of these classes in the following result, see for instance [49], [37], or [50].

Proposition 1.13.

(i) *$A_1 \subseteq A_p \subseteq A_q$, for $1 \leq p \leq q < \infty$.*

(ii) *$RH_\infty \subseteq RH_q \subseteq RH_p$, for $1 < p \leq q \leq \infty$.*

(iii) *If $w \in A_p$, $1 < p < \infty$, then there exists $1 < q < p$ such that $w \in A_q$.*

(iv) *If $w \in RH_s$, $1 < s < \infty$, then there exists $s < r < \infty$ such that $w \in RH_r$.*

(v) $A_\infty = \bigcup_{1\leq p<\infty} A_p = \bigcup_{1<s\leq\infty} RH_s.$

(vi) *If $1 < p < \infty$, $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.*

(vii) *For every $1 < p < \infty$, $w \in A_p$ if and only if $\mathcal{M}$ is bounded on $L^p(w)$. Also, $w \in A_1$ if and only if $\mathcal{M}$ is bounded from $L^1(w)$ into $L^{1,\infty}(w)$.*

For a weight $w \in A_\infty$, define

$$r_w := \inf\{1 \leq r < \infty : w \in A_r\}, \qquad s_w := \inf\{1 \leq s < \infty : w \in RH_{s'}\}. \tag{1.14}$$

Notice that according to our definition s_w is the conjugated exponent of the one defined in [9, Lemma 4.1]. Additionally, we set for any $0 < q < \infty$,

$$q_w^* := \begin{cases} \frac{qnr_w}{nr_w - q} & \text{if} \quad q < nr_w, \\ \infty & \text{if} \quad q \geq nr_w. \end{cases} \tag{1.15}$$

In the case $w \equiv 1$, note that $r_w = 1$ and we write $q^* := q_1^*$.

Given $0 \le p_0 < q_0 \le \infty$, $w \in A_\infty$, and according to [9, Lemma 4.1] we have

$$\mathcal{W}_w(p_0, q_0) := \left\{ p : p_0 < p < q_0, w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'} \right\} = \left(p_0 r_w, \frac{q_0}{s_w} \right). \tag{1.16}$$

If $p_0 = 0$ and $q_0 < \infty$ it is understood that the only condition that stays is $w \in RH_{\left(\frac{q_0}{p}\right)'}$. Analogously, if $0 < p_0$ and $q_0 = \infty$ the only assumption is $w \in A_{\frac{p}{p_0}}$. Finally $\mathcal{W}_w(0, \infty) = (0, \infty)$.

Besides, by [9, Lemma 4.4], we have that

$$p \in \mathcal{W}_w(p_0, q_0) \Leftrightarrow p' \in \mathcal{W}_{w^{1-p'}}(p_0', q_0'). \tag{1.17}$$

Finally, let us also introduce another class of weights that first appeared in [71] to study norm inequalities for fractional integral operators and that we shall use when dealing with Riesz potentials and fractional maximal functions in Section 2.2. Given $1 < p \le q < \infty$, we say that $w \in A_{p,q}$ if for every ball $B \subset \mathbb{R}^n$,

$$\left(\fint_B w(x)^q dx \right)^{\frac{1}{q}} \left(\fint_B w(x)^{-p'} dx \right)^{\frac{1}{p'}} \le C.$$

For $p = 1$, we say that $w \in A_{1,q}$ if for every $x \in B$,

$$\left(\fint_B w(x)^q dx \right)^{\frac{1}{q}} \le C w(x).$$

Note that when $p = q$ the fact that $w \in A_{p,p}$ is equivalent to saying that $w^p \in A_p$. Furthermore, it is easy to see that $w \in A_{p,q}$ if and only if $w \in A_{1+\frac{q}{p'}}$. Hence, the $A_{p,q}$ classes can be seen as subsets of A_∞.

1.2 Elliptic operators

Let A be an $n \times n$ matrix of complex and L^∞-valued coefficients defined on $\mathbb{R}^n$. We assume that this matrix satisfies the following uniform ellipticity (or "accretivity") condition: there exist $0 < \lambda \le \Lambda < \infty$ such that

$$\lambda\, |\xi|^2 \le \operatorname{Re} A(x)\, \xi \cdot \bar{\xi} \qquad \text{and} \qquad |A(x)\, \xi \cdot \bar{\zeta}| \le \Lambda\, |\xi|\, |\zeta|, \tag{1.18}$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. We have used the notation $\xi \cdot \zeta = \xi_1\, \zeta_1 + \cdots + \xi_n\, \zeta_n$ and therefore $\xi \cdot \bar{\zeta}$ is the usual inner product in $\mathbb{C}^n$. Note that then $A(x)\, \xi \cdot \bar{\zeta} = \sum_{j,k} a_{j,k}(x)\, \xi_k\, \bar{\zeta}_j$. Associated with this matrix we define the second order divergence form elliptic operator

$$Lf = -\operatorname{div}(A\, \nabla f), \tag{1.19}$$

which is understood in the standard weak sense as a maximal-accretive operator on $L^2(\mathbb{R}^n, dx)$ with domain $\mathcal{D}(L)$ by means of a sesquilinear form. The operator L has a square root $L^{\frac{1}{2}}$, defined as the unique maximal-accretive operator such that

$$L^{\frac{1}{2}} L^{\frac{1}{2}} = L$$

as unbounded operators (see [3] for a deeper discussion in the operator $L^{\frac{1}{2}}$, and, for a explicit construction, the two references recommended there: [31, Chapter XIV] and [62, p. 281]). We use the following formula to compute $L^{\frac{1}{2}}$:

$$L^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty t L e^{-t^2 L} \frac{dt}{t}. \tag{1.20}$$

Besides, the operator $-L$ generates a C^0-semigroup $\{e^{-tL}\}_{t>0}$ of contractions on $L^2(\mathbb{R}^n)$ which is called the heat semigroup. As in [3] and [10], we denote by $(p_-(L), p_+(L))$ the maximal open interval on which this semigroup, $\{e^{-tL}\}_{t>0}$, is uniformly bounded on $L^p(\mathbb{R}^n)$, and by $(q_-(L), q_+(L))$ the maximal open interval on which the gradient of the heat semigroup, $\{\sqrt{t}\nabla_y e^{-tL}\}_{t>0}$, is uniformly bounded on $L^p(\mathbb{R}^n)$:

$$p_-(L) := \inf\left\{p \in (1,\infty) : \sup_{t>0} \|e^{-t^2L}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty\right\}, \tag{1.21}$$

$$p_+(L) := \sup\left\{p \in (1,\infty) : \sup_{t>0} \|e^{-t^2L}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty\right\}; \tag{1.22}$$

$$q_-(L) := \inf\left\{p \in (1,\infty) : \sup_{t>0} \|t\nabla_y e^{-t^2L}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty\right\}, \tag{1.23}$$

$$q_+(L) := \sup\left\{p \in (1,\infty) : \sup_{t>0} \|t\nabla_y e^{-t^2L}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty\right\}. \tag{1.24}$$

Note that in place of the semigroup $\{e^{-tL}\}_{t>0}$ we are using its rescaling $\{e^{-t^2L}\}_{t>0}$. We do so since all the "heat" square functions, defined below, are written using the latter and also because in the context of the off-diagonal estimates, discussed in the next section, it will simplify some computations.

Furthermore, for every $K \geq 0$, $K \in \mathbb{Q}$ and $0 < q < \infty$, let us set

$$q^{K,*} := \begin{cases} \dfrac{q\,n}{n-(2K+1)\,q}, & \text{if} \quad (2K+1)\,q < n, \\ \infty, & \text{if} \quad (2K+1)\,q \geq n. \end{cases}$$

Corresponding to the case $K = 0$, note that this number is equal to the one defined in (1.15) in the unweighted case. Hence, we write $q^* := q^{0,*}$. We also observe that $q < q^{K,*}$, for all $K \geq 0$, $K \in \mathbb{Q}$.

Besides, from [3] (see also [10]) we know that $p_-(L) = 1$ and $p_+(L) = \infty$ if $n = 1, 2$; and if $n \geq 3$ then $p_-(L) < \frac{2n}{n+2}$ and $p_+(L) > \frac{2n}{n-2}$. Moreover, $q_-(L) = p_-(L)$, $q_+(L)^* \leq p_+(L)$, and we always have $q_+(L) > 2$, with $q_+(L) = \infty$ if $n = 1$.

Following the notation in [11], given a weight $w \in A_\infty$, we also consider the intervals $\widetilde{\mathcal{J}}_w(L)$ and $\widetilde{\mathcal{K}}_w(L)$, which are respectively (possibly empty) intervals of $p \in [1,\infty)$, where $\{e^{-t^2L}\}_{t>0}$ is a bounded set in $\mathcal{L}(L^p(w))$ and $\{t\nabla e^{-t^2L}\}_{t>0}$ is a bounded set in $\mathcal{L}(L^p(w))$ (where $\mathcal{L}(L^p(w))$ denotes the set of linear continuous maps on $L^p(w)$).

Using the heat semigroup and the corresponding Poisson semigroup $\{e^{-t\sqrt{L}}\}_{t>0}$, one can define different conical square functions which all have an expression of the form

$$Q^\alpha f(x) = \left(\iint_{\Gamma^\alpha(x)} |T_t f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}, \qquad x \in \mathbb{R}^n, \tag{1.25}$$

where $\Gamma^\alpha(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t\}$ denotes the cone (of aperture $\alpha > 0$) with vertex at $x \in \mathbb{R}^n$. We just write $Qf(x)$ and $\Gamma(x)$ when $\alpha = 1$. More precisely, we introduce the following conical square functions written in terms of the heat semigroup $\{e^{-tL}\}_{t>0}$ (hence the subscript H): for every $m \in \mathbb{N}$,

$$\mathcal{S}_{m,\mathrm{H}} f(x) = \left(\iint_{\Gamma(x)} |(t^2L)^m e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{1.26}$$

and, for every $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$\mathrm{G}_{m,\mathrm{H}} f(x) = \left(\iint_{\Gamma(x)} |t\nabla_y (t^2L)^m e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{1.27}$$

$$\mathcal{G}_{m,\mathrm{H}}f(x) = \left(\iint_{\Gamma(x)} |t\nabla_{y,t}(t^2L)^m e^{-t^2L}f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}. \tag{1.28}$$

In the same way, let us consider conical square functions associated with the Poisson semigroup $\{e^{-t\sqrt{L}}\}_{t>0}$ (hence the subscript P): given $K \in \mathbb{N}$,

$$\mathcal{S}_{K,\mathrm{P}}f(x) = \left(\iint_{\Gamma(x)} |(t\sqrt{L})^{2K} e^{-t\sqrt{L}}f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{1.29}$$

and for every $K \in \mathbb{N}_0$,

$$\mathrm{G}_{K,\mathrm{P}}f(x) = \left(\iint_{\Gamma(x)} |t\nabla_y(t\sqrt{L})^{2K} e^{-t\sqrt{L}}f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{1.30}$$

$$\mathcal{G}_{K,\mathrm{P}}f(x) = \left(\iint_{\Gamma(x)} |t\nabla_{y,t}(t\sqrt{L})^{2K} e^{-t\sqrt{L}}f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}. \tag{1.31}$$

Corresponding to the cases $m = 0$ or $K = 0$ we simply write $\mathrm{G}_{\mathrm{H}}f := \mathrm{G}_{0,\mathrm{H}}f$, $\mathcal{G}_{\mathrm{H}}f := \mathcal{G}_{0,\mathrm{H}}f$, $\mathrm{G}_{\mathrm{P}}f := \mathrm{G}_{0,\mathrm{P}}f$, and $\mathcal{G}_{\mathrm{P}}f := \mathcal{G}_{0,\mathrm{P}}f$. Besides, we set $\mathcal{S}_{\mathrm{H}}f := \mathcal{S}_{1,\mathrm{H}}f$ and $\mathcal{S}_{\mathrm{P}}f := \mathcal{S}_{1,\mathrm{P}}f$.

We also consider the following non-tangential maximal functions.

$$\mathcal{N}_{\mathrm{H}}f(x) = \sup_{(y,t)\in\Gamma(x)} \left(\int_{B(y,t)} |e^{-t^2L}f(z)|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}} \quad \text{and} \quad \mathcal{N}_{\mathrm{P}}f(x) = \sup_{(y,t)\in\Gamma(x)} \left(\int_{B(y,t)} |e^{-t\sqrt{L}}f(z)|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}}. \tag{1.32}$$

Finally, we denote the Riesz transform associated with the operator L by $\nabla L^{-\frac{1}{2}}$. This operator has the following integral representation:

$$\nabla L^{-\frac{1}{2}}h = \frac{2}{\sqrt{\pi}} \int_0^\infty t\nabla e^{-t^2L}h \frac{dt}{t}. \tag{1.33}$$

From [11] we know the following boundedness result for the Riesz transform.

Theorem 1.34. *[11, Theorem 5.2] Let $w \in A_\infty$ be such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. For all $f \in L_c^\infty(\mathbb{R}^n)$ and $p \in \mathrm{Int}\widetilde{\mathcal{K}}_{\mathrm{w}}(\mathrm{L})$,*

$$\|\nabla L^{-\frac{1}{2}}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Hence $\nabla L^{-\frac{1}{2}}$ has a bounded extension to $L^p(w)$.

1.3 Off-diagonal estimates

1.3.1 Off-diagonal estimate in $\mathbb{R}^n$

We briefly recall the notion of off-diagonal estimates. Let $\{T_t\}_{t>0}$ be a family of linear operators and let $1 \le p \le q \le \infty$. We say that $\{T_t\}_{t>0}$ satisfies $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ off-diagonal estimates of exponential type, denoted by $\{T_t\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$, if for all closed sets E, F, all f, and all $t > 0$ we have

$$\|T_t(f\,\mathbf{1}_E)\,\mathbf{1}_F\|_{L^q(\mathbb{R}^n)} \le Ct^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-c\frac{d(E,F)^2}{t^2}} \|f\,\mathbf{1}_E\|_{L^p(\mathbb{R}^n)}.$$

Analogously, given $0 < \beta < \infty$, we say that $\{T_t\}_{t>0}$ satisfies $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ off-diagonal estimates of polynomial type with order $0 < \beta < \infty$, denoted by $\{T_t\}_{t>0} \in \mathcal{F}_\beta(L^p \to L^q)$ if for all closed sets E, F, all f, and all $t > 0$ we have

$$\|T_t(f\,\mathbf{1}_E)\,\mathbf{1}_F\|_{L^q(\mathbb{R}^n)} \le Ct^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left(1 + \frac{d(E,F)^2}{t^2}\right)^{-\left(\beta+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \|f\,\mathbf{1}_E\|_{L^p(\mathbb{R}^n)}.$$

The heat and the Poisson semigroups satisfy respectively off-diagonal estimates of exponential and polynomial type. Before making this precise, let us recall the definition of $p_-(L)$ and $p_+(L)$ in (1.21)-(1.22), and that of $q_-(L)$ and $q_+(L)$ in (1.23)-(1.24). The importance of these parameters stems from the fact that, besides giving the maximal intervals on which either the heat semigroup or its gradient is uniformly bounded, they characterize the maximal open intervals on which off-diagonal estimates of exponential type hold (see [3] and [10]). More precisely, for every $m \in \mathbb{N}_0$, there hold

$$\{(t^2L)^m e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q) \quad \text{for all} \quad p_-(L) < p \le q < p_+(L)$$

and

$$\{t\nabla_y e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q) \quad \text{for all} \quad q_-(L) < p \le q < q_+(L).$$

From these off-diagonal estimates we shall show that, for every $m \in \mathbb{N}_0$,

$$\{(t\sqrt{L})^{2m} e^{-t\sqrt{L}}\}_{t>0}, \in \mathcal{F}_{m+\frac{1}{2}}(L^p \to L^q),$$

for all $p_-(L) < p \le q < p_+(L)$, and

$$\{t\nabla_y (t^2L)^m e^{-t^2L}\}_{t>0},\ \{t\nabla_{y,t} (t^2L)^m e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q),$$

$$\{t\nabla_y (t\sqrt{L})^{2m} e^{-t\sqrt{L}}\}_{t>0} \in \mathcal{F}_{m+1}(L^p \to L^q),\ \{t\nabla_{y,t} (t\sqrt{L})^{2m} e^{-t\sqrt{L}}\}_{t>0} \in \mathcal{F}_{m+\frac{1}{2}}(L^p \to L^q),$$

for all $q_-(L) < p \le q < q_+(L)$.

To show these off-diagonal estimates we will apply the following Lemma, whose proof follows *mutatis mutandis* from [54, Lemma 2.3].

Lemma 1.35. *Let $\{P_t\}_{t>0}$ and $\{Q_s\}_{s>0}$ be two families of linear operators. Given $1 \le p \le q \le \infty$, assume that $\{P_t\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$ and $\{Q_s\}_{s>0} \in \mathcal{F}_\infty(L^p \to L^p)$. Then, for all closed sets E, F, all f, and all $t, s > 0$ we have*

$$\left\|(P_t \circ Q_s)(f\,\mathbf{1}_E)\,\mathbf{1}_F\right\|_{L^q(\mathbb{R}^n)} \le C t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-c\frac{d(E,F)^2}{\max\{t,s\}^2}} \|f\,\mathbf{1}_E\|_{L^p(\mathbb{R}^n)}.$$

To prove our claims, let us first consider

$$t\nabla_y (t^2L)^m e^{-t^2L} = C\frac{t}{\sqrt{2}}\nabla_y e^{-\frac{t^2}{2}L} \circ \left(\frac{t^2}{2}L\right)^m e^{-\frac{t^2}{2}L}.$$

Taking $P_t = \frac{t}{\sqrt{2}}\nabla_y e^{-\frac{t^2}{2}L}$ and $Q_t = \left(\frac{t^2}{2}L\right)^m e^{-\frac{t^2}{2}L}$ for all $t > 0$, since $\{P_t\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$ and $\{Q_t\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^p)$, for all $q_-(L) < p \le q < q_+(L)$, we conclude from Lemma 1.35 that $\{t\nabla_y (t^2L)^m e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$, for all $q_-(L) < p \le q < q_+(L)$.

To prove that $\{t\nabla_{y,t} (t^2L)^m e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$, for all $q_-(L) < p \le q < q_+(L)$, we just need to observe that

$$|t\nabla_{y,t} (t^2L)^m e^{-t^2L} f(y)| \lesssim |t\nabla_y (t^2L)^m e^{-t^2L} f(y)| + |(t^2L)^m e^{-t^2L} f(y)| + |(t^2L)^{m+1} e^{-t^2L} f(y)|,$$

and apply the off-diagonal estimates satisfied by each term.

We next obtain the off-diagonal estimates of polynomial type satisfied by the operators related to the Poisson semigroup. Following some ideas used in [55, Lemma 5.1], we shall combine the subordination formula

$$e^{-t\sqrt{L}} f(y) = C\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2L}{4u}} f(y)\, du \tag{1.36}$$

with the off-diagonal estimates satisfied by $\{(t^2L)^m e^{-t^2L}\}_{t>0}$ and by $\{t\nabla_y (t^2L)^m e^{-t^2L}\}_{t>0}$ and Minkowski's inequality.

To obtain that $\{(t\sqrt{L})^{2m} e^{-t\sqrt{L}}\}_{t>0} \in \mathcal{F}_{m+\frac{1}{2}}(L^p \to L^q)$ for all $p_-(L) < p \le q < p_+(L)$, take two closed sets E and F, a function f supported in E, and $t > 0$. We apply (1.36), Minkowski's inequality, the off-diagonal estimates satisfied by $\{(tL)^m e^{-tL}\}_{t>0}$, and change the variable u into $\left(1 + d(E,F)^2/t^2\right)^{-1} u$:

$$\begin{aligned}
\left(\int_F |(t\sqrt{L})^{2m} e^{-t\sqrt{L}} f(y)|^q\,dy\right)^{\frac{1}{q}} &= C\left(\int_F \left|(t\sqrt{L})^{2m}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2}{4u}L} f(y)\,du\right|^q dy\right)^{\frac{1}{q}}\\
&\lesssim \int_0^\infty e^{-u} u^{m+\frac{1}{2}} \left(\int_F \left|\left(\frac{t^2}{4u}L\right)^m e^{-\frac{t^2}{4u}L} f(y)\right|^q dy\right)^{\frac{1}{q}} \frac{du}{u}\\
&\lesssim \int_0^\infty e^{-cu\left(1+\frac{d(E,F)^2}{t^2}\right)} u^{m+\frac{1}{2}} \left(\frac{t}{\sqrt{u}}\right)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \frac{du}{u} \left(\int_E |f(y)|^p\,dy\right)^{\frac{1}{p}}\\
&= Ct^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left(1+\frac{d(E,F)^2}{t^2}\right)^{-\left(m+\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \int_0^\infty e^{-cu} u^{m+\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \frac{du}{u}\left(\int_E |f(y)|^p\,dy\right)^{\frac{1}{p}}\\
&= C\,t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left(1+\frac{d(E,F)^2}{t^2}\right)^{-\left(m+\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \left(\int_E |f(y)|^p\,dy\right)^{\frac{1}{p}},
\end{aligned}$$

where in the last equality we have used that $m \ge 0$ and that $p \le q$.

We next show that $\{t\nabla_{y,t}(t\sqrt{L})^{2m} e^{-t\sqrt{L}}\}_{t>0} \in \mathcal{F}_{m+\frac{1}{2}}(L^p \to L^q)$ for all $q_-(L) < p \le q < q_+(L)$. By applying the subordination formula (1.36), and Minkowski's inequality, we obtain

$$\begin{aligned}
\left(\int_F |t\nabla_{y,t}(t\sqrt{L})^{2m} e^{-t\sqrt{L}} f(y)|^q\,dy\right)^{\frac{1}{q}} &= C\left(\int_F \left|t\nabla_{y,t}(t\sqrt{L})^{2m}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2}{4u}L} f(y)\,du\right|^q dy\right)^{\frac{1}{q}}\\
&\le C\int_0^\infty e^{-u} u^{m+1}\left(\int_F \left|\frac{t}{2\sqrt{u}}\nabla_{y,t}\left(\frac{t}{2\sqrt{u}}\sqrt{L}\right)^{2m} e^{-\frac{t^2}{4u}L} f(y)\right|^q dy\right)^{\frac{1}{q}} \frac{du}{u}.
\end{aligned}$$

Note now that

$$\begin{aligned}
\left|\frac{t}{2\sqrt{u}}\nabla_{y,t}\left(\frac{t}{2\sqrt{u}}\sqrt{L}\right)^{2m} e^{-\frac{t^2}{4u}L} f(y)\right| &\approx \left|\frac{t}{2\sqrt{u}}\nabla_{y}\left(\frac{t}{2\sqrt{u}}\sqrt{L}\right)^{2m} e^{-\frac{t^2}{4u}L} f(y)\right|\\
&\quad + \left|u^{-\frac{1}{2}}\left(\frac{t}{2\sqrt{u}}\sqrt{L}\right)^{2m} e^{-\frac{t^2}{4u}L} f(y)\right| + \left|u^{-\frac{1}{2}}\left(\frac{t}{2\sqrt{u}}\sqrt{L}\right)^{2(m+1)} e^{-\frac{t^2}{4u}L} f(y)\right|.
\end{aligned}$$

Then, applying that, for all $K \in \mathbb{N}_0$, $\{t\nabla_y(t\sqrt{L})^{2K} e^{-t^2L}\}_{t>0}, \{(t^2L)^K e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$, we have

$$\begin{aligned}
&\left(\int_F |t\nabla_{y,t}(t\sqrt{L})^{2m} e^{-t\sqrt{L}} f(y)|^q\,dy\right)^{\frac{1}{q}}\\
&\quad\lesssim t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\int_0^\infty \left(u^{m+1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} + u^{m+\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right) e^{-c\left(1+\frac{d(E,F)^2}{t^2}\right)u} \frac{du}{u}\, \|f\|_{L^p(E)}\\
&\quad\le C\,t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left(1+\frac{d(E,F)^2}{t^2}\right)^{-\left(m+\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \|f\|_{L^p(E)}.
\end{aligned}$$

Finally, to show that $\{t\nabla_y(t\sqrt{L})^{2m} e^{-t\sqrt{L}}\}_{t>0} \in \mathcal{F}_{m+1}(L^p \to L^q)$ we proceed as above. Using the fact that, for all $K \in \mathbb{N}_0$, $\{t\nabla_y(t\sqrt{L})^{2K} e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$, we have

$$\left(\int_F |t\nabla_y(t\sqrt{L})^{2m} e^{-t\sqrt{L}} f(y)|^q\,dy\right)^{\frac{1}{q}} = C\left(\int_F \left|t\nabla_y(t\sqrt{L})^{2m}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2}{4u}L} f(y)\,du\right|^q dy\right)^{\frac{1}{q}}$$

$$\leq C \int_0^\infty e^{-u} u^{m+1} \left(\int_F \left| \frac{t}{2\sqrt{u}} \nabla_y \left(\frac{t}{2\sqrt{u}} \sqrt{L} \right)^{2m} e^{-\frac{t^2}{4u}L} f(y) \right|^q dy \right)^{\frac{1}{q}} \frac{du}{u}.$$

$$\lesssim t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \int_0^\infty u^{m+1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-c\left(1+\frac{d(E,F)^2}{t^2}\right)u} \frac{du}{u} \|f\|_{L^p(E)}$$

$$\leq C\, t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left(1 + \frac{d(E,F)^2}{t^2}\right)^{-\left(m+1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \|f\|_{L^p(E)}.$$

Now, consider the family $\{\mathcal{T}_{t,s}\}_{s,t>0}$, where $\mathcal{T}_{t,s} := (e^{-t^2L} - e^{-(t^2+s^2)L})^M$, for an arbitrary $M \in \mathbb{N}$. This family will appear naturally in several proofs in Chapters 3 and 4, and it will be very convenient to know what type of off-diagonal estimates it satisfies.

Proposition 1.37. *For $0 < t, s < \infty$, $p, q \in (p_-(L), p_+(L))$, $p \leq q$, $M \in \mathbb{N}$, and for E_1, E_2 closed subsets in $\mathbb{R}^n$, and $f \in L^p(\mathbb{R}^n)$ such that $supp(f) \subset E_1$, we have that $\{\mathcal{T}_{t,s}\}_{s,t>0}$ satisfies the following $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ off-diagonal estimates:*

$$\left\|\mathcal{T}_{t,s} f\right\|_{L^q(E_2)} \lesssim \left(\frac{s^2}{t^2}\right)^M t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} e^{-c\frac{d(E_1,E_2)^2}{t^2+s^2}} \|f\|_{L^p(E_1)}, \tag{1.38}$$

Proof. Note that we have

$$\left\|\mathcal{T}_{t,s} f\right\|_{L^q(E_2)} = \left\|(e^{-t^2L} - e^{-(t^2+s^2)L})^M f\right\|_{L^q(E_2)} = \left\| \left(\int_0^{s^2} \partial_r e^{-(r+t^2)L} \, dr \right)^M f \right\|_{L^q(E_2)}$$

$$\leq \int_0^{s^2} \cdots \int_0^{s^2} \left\| \left((r_1 + \cdots + r_M + Mt^2)L\right)^M e^{-(r_1+\cdots+r_M+Mt^2)L} f \right\|_{L^q(E_2)} \frac{dr_1 \cdots dr_M}{(r_1 + \cdots + r_M + Mt^2)^M}$$

$$\lesssim \int_0^{s^2} \cdots \int_0^{s^2} e^{-c\frac{d(E_1,E_2)^2}{r_1+\cdots+r_M+Mt^2}} \frac{dr_1 \cdots dr_M}{(r_1 + \cdots + r_M + Mt^2)^M} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(E_1)}$$

$$\lesssim \left(\frac{s^2}{t^2}\right)^M e^{-c\frac{d(E_1,E_2)^2}{t^2+s^2}} t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(E_1)},$$

where we have used that $\{(t^2L)^M e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$. □

1.3.2 Weighted off-diagonal estimates on balls

We denote by $(\mathcal{X}, d, \mu)$ a space of homogeneous type. This is, a non-empty set endowed with a distance or a quasi-distance d and a non-negative Borel measure μ on $\mathcal{X}$ satisfying the doubling condition: there exists $C_0 > 0$ such that

$$\mu(B(x, 2r)) \leq C_0\, \mu(B(x, r)) < \infty,$$

for all $x \in \mathcal{X}$, $r > 0$, and where $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$. In [10] the authors gave the definition of off-diagonal estimates on balls on spaces of homogeneous type. In particular, if we consider $\mathbb{R}^n$ with the Euclidean distance and the measure given by a weight, which we know is a doubling measure in $\mathbb{R}^n$ (see (1.11)), we have a space of homogeneous type, and then we can write the definition of weighted off-diagonal estimates on balls as follows.

Definition 1.39. *[11, Definition 3.2] Given $1 \leq p \leq q \leq \infty$ and any weight $w \in A_\infty$, we say that a family of sublinear operators $\{T_t\}_{t>0}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls, in short $T_t \in \mathcal{O}(L^p(w) - L^q(w))$, if there exist $\theta_1, \theta_2 > 0$ and $c > 0$ such that for every $t > 0$ and for any ball B with radius r_B and all f,*

$$\left(\fint_B |T_t(f\mathbf{1}_B)(x)|^q dw \right)^{\frac{1}{q}} \lesssim \Upsilon\left(\frac{r_B}{\sqrt{t}}\right)^{\theta_2} \left(\fint_B |f(x)|^p dw \right)^{\frac{1}{p}};$$

and, for all $j \geq 2$,

$$\left(\fint_B |T_t(f\mathbf{1}_{C_j(B)})(x)|^q dw\right)^{\frac{1}{q}} \lesssim 2^{j\theta_1}\Upsilon\left(\frac{2^j r_B}{\sqrt{t}}\right)^{\theta_2} e^{-c\frac{4^j r_B^2}{t}} \left(\fint_{C_j(B)} |f(x)|^p dw\right)^{\frac{1}{p}}$$

and

$$\left(\fint_{C_j(B)} |T_t(f\mathbf{1}_B)(x)|^q dw\right)^{\frac{1}{q}} \lesssim 2^{j\theta_1}\Upsilon\left(\frac{2^j r_B}{\sqrt{t}}\right)^{\theta_2} e^{-c\frac{4^j r_B^2}{t}} \left(\fint_B |f(x)|^p dw\right)^{\frac{1}{p}}.$$

Where $\Upsilon(s) := \max\{s, s^{-1}\}$ and $C_j(B)$ are the annuli of B (see Notation (o)*).*

We have that when $w = 1$, $L^p(w) - L^q(w)$ off-diagonal estimates on balls are equivalent to $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ usual off-diagonal estimates of exponential type, defined in the previous section.

Remark 1.40. *In order to avoid confusion, we observe that in the context of usual off-diagonal estimates (the ones defined in the previous section), we consider the parameter of the families at scale t^2, while in the context of weighted off-diagonal estimates on balls the parameter of the families is at scale t. For instance, for $p < q$ in the appropriated range, we have that $\{e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^p \to L^q)$ while in the case of weighted off-diagonal estimates on balls the correct formulation is $\{e^{-tL}\}_{t>0} \in \mathcal{O}(L^p(w) - L^q(w))$. We do so for convenience.*

The families associated with the operator L that we presented in the previous section satisfy off-diagonal estimates on balls. The result, that appears in [10] and in [11, Proposition 3.4] in a more general setting, is the following:

Proposition 1.41. *Given $w \in A_\infty$ and $m \in \mathbb{N}_0$*

(*a*) *Assume that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$. There exists a maximal interval of $[1, \infty]$, denoted by $\mathcal{J}_w(L)$, containing $\mathcal{W}_w(p_-(L), p_+(L))$, such that if $p, q \in \mathcal{J}_w(L)$ with $p \leq q$, then $\{(tL)^m e^{-tL}\}_{t>0} \in \mathcal{O}(L^p(w) - L^q(w))$ and is a bounded set in $\mathcal{L}(L^p(w))$. Furthermore, $\mathcal{J}_w(L) \subset \widetilde{\mathcal{J}}_w(L)$ and $Int\mathcal{J}_w(L) = Int\widetilde{\mathcal{J}}_w(L)$.*

(*b*) *Assume that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. There exists a maximal interval of $[1, \infty]$, denoted by $\mathcal{K}_w(L)$, containing $\mathcal{W}_w(q_-(L), q_+(L))$, such that if $p, q \in \mathcal{K}_w(L)$ with $p \leq q$, then $\{\sqrt{t}\nabla_y(tL)^m e^{-tL}\}_{t>0} \in \mathcal{O}(L^p(w) - L^q(w))$ and is a bounded set in $\mathcal{L}(L^p(w))$. Furthermore, $\mathcal{K}_w(L) \subset \widetilde{\mathcal{K}}_w(L)$ and $Int\mathcal{K}_w(L) = Int\widetilde{\mathcal{K}}_w(L)$.*

(*c*) *Let $n \geq 2$. Assume $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. Then $\mathcal{K}_w(L) \subset \mathcal{J}_w(L)$. Moreover, $\inf \mathcal{J}_w(L) = \inf \mathcal{K}_w(L)$ and $(\sup \mathcal{K}_w(L))^*_w \leq \sup \mathcal{J}_w(L)$ (recall the definition of q^*_w, for any $0 < q < \infty$, in* (1.15)*).*

(*d*) *If $n = 1$, the intervals $\mathcal{J}_w(L)$ and $\mathcal{K}_w(L)$ are the same and contain $(r_w, \infty]$ if $w \notin A_1$ and are equal to $[1, \infty]$ if $w \in A_1$.*

Recall the definitions of $\widetilde{\mathcal{J}}_w(L)$ and $\widetilde{\mathcal{K}}_w(L)$ on page 5.

Remark 1.42. *In [11] the authors observed that if L is a real operator then $\mathcal{W}_w(p_-(L), p_+(L)) = Int\mathcal{J}_w(L)$. However, in the case of complex operators we do not know whether $\mathcal{J}_w(L)$ and $\mathcal{W}_w(p_-(L), p_+(L))$ have different end-points. Hereafter, we denote $Int\mathcal{J}_w(L) = (\widehat{p}_-(L), \widehat{p}_+(L))$.*

Finally, as in the previous section, consider the family $\{\mathcal{T}_{t,s}\}_{s,t>0}$, where $\mathcal{T}_{t,s} := (e^{-t^2L} - e^{-(t^2+s^2)L})^M$, for an arbitrary $M \in \mathbb{N}$. As we explained, this family will appear naturally in several proofs of Chapters 3 and 4, and it will be also very convenient to know what type of weighted off-diagonal estimates on balls it satisfies.

Proposition 1.43. *Given $w \in A_\infty$ such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, for $0 < t, s < \infty$, $p, q \in \mathcal{J}_w(L)$, $p \leq q$, $M \in \mathbb{N}$, and a ball $B \subset \mathbb{R}^n$, we have that $\{\mathcal{T}_{t,s}\}_{s,t>0}$ satisfies the following $L^p(w) - L^q(w)$ off-diagonal estimates on balls: there exist $\theta_1, \theta_2 > 0$ such that, for all $j \geq 2$,*

$$\left(\fint_{C_j(B)} |\mathcal{T}_{t,s}(f\mathbf{1}_B)(x)|^q dw \right)^{\frac{1}{q}} \lesssim 2^{j\theta_1} \max\left\{ \frac{2^j r_B}{t}, \frac{\sqrt{s^2+t^2}}{2^j r_B} \right\}^{\theta_2} \left(\frac{s^2}{t^2}\right)^M e^{-c\frac{4^j r_B^2}{t^2+s^2}} \left(\fint_B |f(x)|^p dw \right)^{\frac{1}{p}}; \tag{1.44}$$

and,

$$\left(\fint_B |\mathcal{T}_{t,s}(f\mathbf{1}_B)(x)|^q dw \right)^{\frac{1}{q}} \lesssim \max\left\{ \frac{r_B}{t}, \frac{\sqrt{s^2+t^2}}{r_B} \right\}^{\theta_2} \left(\frac{s^2}{t^2}\right)^M \left(\fint_B |f(x)|^p dw \right)^{\frac{1}{p}}. \tag{1.45}$$

Proof. As in the proof of Proposition 1.37, we have

$$\begin{aligned}
&\left(\fint_B |\mathcal{T}_{t,s}(f\mathbf{1}_B)(x)|^q dw \right)^{\frac{1}{q}} = w(2^{j+1}B)^{-\frac{1}{q}} \left\| \left(\left(\int_0^{s^2} \partial_r e^{-(r+t^2)L}\, dr \right)^M f\mathbf{1}_B \right) \mathbf{1}_{C_j(B)} \right\|_{L^q(w)} \\
&\leq w(2^{j+1}B)^{-\frac{1}{q}} \\
&\quad \int_0^{s^2} \cdots \int_0^{s^2} \left\| \left(\left((r_1 + \cdots + r_M + Mt^2)L \right)^M e^{-(r_1+\cdots+r_M+Mt^2)L} (f\mathbf{1}_B) \right) \mathbf{1}_{C_j(B)} \right\|_{L^q(w)} \frac{dr_1 \cdots dr_M}{(r_1 + \cdots + r_M + Mt^2)^M} \\
&\lesssim 2^{j\theta_1} \int_0^{s^2} \cdots \int_0^{s^2} \Upsilon\left(\frac{2^j r_B}{\sqrt{r_1 + \cdots + r_M + Mt^2}} \right)^{\theta_2} e^{-c\frac{4^j r_B^2}{r_1+\cdots+r_M+Mt^2}} \frac{dr_1 \cdots dr_M}{(r_1 + \cdots + r_M + Mt^2)^M} w(B)^{-\frac{1}{p}} \|f\mathbf{1}_B\|_{L^p(w)} \\
&\lesssim 2^{j\theta_1} \max\left\{ \frac{2^j r_B}{t}, \frac{\sqrt{s^2+t^2}}{2^j r_B} \right\}^{\theta_2} \left(\frac{s^2}{t^2}\right)^M e^{-c\frac{4^j r_B^2}{t^2+s^2}} \left(\fint_B |f(x)|^p dw \right)^{\frac{1}{p}},
\end{aligned}$$

where $\theta_1, \theta_2 > 0$, and we have used the fact that the family $\{tLe^{-tL}\}_{t>0}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls (see Proposition 1.41).

The estimate of (1.45) follows similarly. □

1.4 Extrapolation

J. L. Rubio de Francia proved in [73] that if an operator T is bounded on $L^p(w)$ for some $1 \leq p < \infty$ and for all $w \in A_p$, then T is bounded on $L^q(w)$ for every $w \in A_q$ and $1 < q < \infty$. This result is known as the Rubio de Francia extrapolation theorem. Later, in [47] J. L. García-Cuerva gave a new proof of this result independent of vector valued inequalities. After this, the extrapolation theory developed and proved to be a very useful tool in harmonic analysis, and in particular in the work we present here. Specifically, we use the following results:

Theorem 1.46. *Let $\mathcal{F}$ be a given family of pairs (f, g) of non-negative and not identically zero measurable functions.*

(*a*) *Suppose that for some fixed exponent p_0, $1 \leq p_0 < \infty$, and every weight $w \in A_{p_0}$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \leq C_w \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad \forall\, (f, g) \in \mathcal{F}. \tag{1.47}$$

Then, for all $1 < p < \infty$, and for all $w \in A_p$,

$$\int_{\mathbb{R}^n} f(x)^p\, w(x)\, dx \leq C_{w,p} \int_{\mathbb{R}^n} g(x)^p\, w(x)\, dx, \qquad \forall\, (f, g) \in \mathcal{F}. \tag{1.48}$$

(*b*) *Suppose that for some fixed exponent q_0, $1 \le q_0 < \infty$, and every weight $w \in RH_{q_0'}$,*

$$\int_{\mathbb{R}^n} f(x)^{\frac{1}{q_0}}\, w(x)\, dx \le C_w \int_{\mathbb{R}^n} g(x)^{\frac{1}{q_0}}\, w(x)\, dx, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.49}$$

Then, for all $1 < q < \infty$ and for all $w \in RH_{q'}$,

$$\int_{\mathbb{R}^n} f(x)^{\frac{1}{q}}\, w(x)\, dx \le C_{w,q} \int_{\mathbb{R}^n} g(x)^{\frac{1}{q}}\, w(x)\, dx, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.50}$$

(*c*) *Let $1 \le p_- < p_+ < \infty$. Suppose that there exists p_0, $p_- \le p_0 \le p_+$, such that for every weight $w \in A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'}$*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \le C_{w,q} \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.51}$$

Then, for all p, $p_- < p < p_+$, and for all $w \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'}$

$$\int_{\mathbb{R}^n} f(x)^{p}\, w(x)\, dx \le C_{w,q} \int_{\mathbb{R}^n} g(x)^{p}\, w(x)\, dx, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.52}$$

(*d*) *Suppose that for some p_0, q_0, $1 \le p_0 \le q_0 < \infty$, and every weight $w \in A_{p_0,q_0}$*

$$\left(\int_{\mathbb{R}^n} f(x)^{q_0}\, w(x)^{q_0}\, dx\right)^{\frac{1}{q_0}} \le C_{w,q} \left(\int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)^{p_0}\, dx\right)^{\frac{1}{p_0}}, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.53}$$

Then, for all p and q such that $1 < p \le q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and for all $w \in A_{p,q}$,

$$\left(\int_{\mathbb{R}^n} f(x)^{q}\, w(x)^{q}\, dx\right)^{\frac{1}{q}} \le C_{w,q} \left(\int_{\mathbb{R}^n} g(x)^{p}\, w(x)^{p}\, dx\right)^{\frac{1}{p}}, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.54}$$

(*e*) *Suppose that for some fixed exponent r_0, $0 < r_0 < \infty$, and every weight $w \in A_\infty$*

$$\int_{\mathbb{R}^n} f(x)^{r_0}\, w(x)\, dx \le C_w \int_{\mathbb{R}^n} g(x)^{r_0}\, w(x)\, dx, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.55}$$

Then, for all $0 < r < \infty$, and for all $w \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^{r}\, w(x)\, dx \le C_{w,r} \int_{\mathbb{R}^n} g(x)^{r}\, w(x)\, dx, \qquad \forall\, (f,g) \in \mathcal{F}. \tag{1.56}$$

Note that part (a) is the Rubio de Francia extrapolation theorem, written in terms of pairs of functions rather than in terms of boundedness of operators. The reader is referred to [35] for a complete account of this topic. We can find part (a), part (c), part (d), and part (e) in [35] as Theorems 3.9, 3.31, and 3.23, and Corollary 3.15, respectively. However, there is a subtle difference between the results present here and the results in [35]: in the latter both the hypothesis and the conclusions are assumed to hold for all pairs $(f, g) \in \mathcal{F}$ for which the left-hand sides are finite. Here we do not make such assumptions and, in particular, we do have that the infiniteness of the left-hand side will imply that of the right-hand one. This formulation is more convenient for our purposes and its proof becomes a simple consequence of [35, Theorems 3.9, 3.31, and 3.23, and Corollary 3.15]. The extrapolation result in (b) is not written explicitly in [35], but can be easily obtained using [9, Theorem 4.9] and [35, Theorem 3.31] (see also [7, Proposition 2.3] for a particular case).

Proof. We first prove (*a*), (*c*), (*d*), and (*e*) in the present form (that is without assuming that the left-hand sides are finite) follow easily from the corresponding results in [35]. Given a family $\mathcal{F}$ as in the statement and an arbitrary large number $N > 0$ we consider the new family

$$\mathcal{F}_N := \left\{(f_N, g) : (f, g) \in \mathcal{F}, f_N := f\mathbf{1}_{\{x\in B(0,N): f(x)\le N\}}\right\}.$$

Note that

$$\int_{\mathbb{R}^n} f_N(x)^r w(x)dx \le N^r w(B(0,N)) < \infty, \quad \text{for all} \quad 0 < r < \infty \quad \text{and} \quad w \in A_\infty. \tag{1.57}$$

From (1.47), (1.51), and (1.53), and the fact that $f_N \le f$, we clearly obtain that the same estimates hold for every pair in $\mathcal{F}_N$ (with a constant uniform in N) with a left-hand side that is always finite by (1.57). Thus we can apply [35, Theorems 3.9, 3.31, and 3.23, and Corollary 3.15] to $\mathcal{F}_N$ to conclude that (1.48), (1.52), and (1.54) hold for all pairs $(f_N, g) \in \mathcal{F}_N$ (with a constant uniform in N), since again the left-hand side is always finite by (1.57). To complete the proof we just need to invoke the Monotone Convergence Theorem.

We finally obtain (*b*). Let us fix $1 < q < \infty$ and $w \in RH_{q'}$. As before we first work with $\mathcal{F}_N$. Since $w \in RH_{q'} \subset A_\infty$, there exists p_0 such that $w \in A_{p_0}$. We set $p_+ := 2q$, $r_0 := \frac{2q}{q_0}$ and pick $0 < p_- < \min\left\{\frac{2q}{q_0}, \frac{2}{p_0}, 2\right\}$. We then have that $0 < p_- < r_0 \le p_+$, and for all $w_0 \in A_{\frac{r_0}{p_-}} \cap RH_{\left(\frac{p_+}{r_0}\right)'} \subset RH_{\left(\frac{p_+}{r_0}\right)'} = RH_{q_0'}$,

$$\begin{aligned}\int_{\mathbb{R}^n} \left(f_N(x)^{\frac{1}{2q}}\right)^{r_0} w_0(x)dx &= \int_{\mathbb{R}^n} f_N(x)^{\frac{1}{q_0}} w_0(x)dx \le \int_{\mathbb{R}^n} f(x)^{\frac{1}{q_0}} w_0(x)dx \\ &\le C\int_{\mathbb{R}^n} g(x)^{\frac{1}{q_0}} w_0(x)dx = C\int_{\mathbb{R}^n} \left(g(x)^{\frac{1}{2q}}\right)^{r_0} w_0(x)dx,\end{aligned} \tag{1.58}$$

with C independent of N, and for every pair $(f_N, g) \in \mathcal{F}_N$. Note that for each pair the left-hand side is finite by (1.57). Therefore, applying [9, Theorem 4.9] or [35, Theorem 3.31], we obtain, for all $p_- < p < p_+$ and for all $\widetilde{w} \in A_{\frac{p}{p_-}} \cap RH_{\left(\frac{p_+}{p}\right)'}$,

$$\int_{\mathbb{R}^n} f_N(x)^{\frac{p}{2q}} \widetilde{w}(x)dx \le C\int_{\mathbb{R}^n} g(x)^{\frac{p}{2q}} \widetilde{w}(x)dx, \tag{1.59}$$

with C independent of N, for every pair $(f_N, g) \in \mathcal{F}_N$. Then, note that $p = 2$ satisfies that $p_- < p < p_+$ and also $w \in A_{p_0} \cap RH_{q'} \subset A_{\frac{2}{p_-}} \cap RH_{(\frac{p_+}{2})'}$. Thus, we can apply (1.59) with $p = 2$ and $\widetilde{w} = w$ to obtain

$$\int_{\mathbb{R}^n} f_N(x)^{\frac{1}{q}} w(x)dx \le C\int_{\mathbb{R}^n} g(x)^{\frac{1}{q}} w(x)dx,$$

with C independent of N. Letting $N \to \infty$, the Monotone Convergence Theorem yields the desired estimate (1.50). □

Chapter 2
TENT SPACES

Tent spaces were introduced in [32] by R. R. Coifman, Y. Meyer, and E. M. Stein. These spaces are related to Hardy and L^p spaces and have proved to be a very useful tool in harmonic analysis, for instance in the study of square and maximal functions, or PDE's theory.

2.1 Tent spaces in $\mathbb{R}^n$

Tent spaces are defined as follows. Let $\mathbb{R}^{n+1}_+$ denote the upper-half space, that is, the set of points $(y,t) \in \mathbb{R}^n \times \mathbb{R}$ with $t > 0$. Given $\alpha > 0$ and $x \in \mathbb{R}^n$ we define the cone of aperture α with vertex at x by

$$\Gamma^\alpha(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t\},$$

when $\alpha = 1$ we simply write $\Gamma(x)$. For a closed set E in $\mathbb{R}^n$, we denote

$$\mathcal{R}^\alpha(E) := \bigcup_{x \in E} \Gamma^\alpha(x).$$

When $\alpha = 1$ we simplify the notation by writing $\mathcal{R}(E)$ instead of $\mathcal{R}^1(E)$.

For $\alpha > 0$, consider, for $0 < r < \infty$, the operator $\mathcal{A}^\alpha_r$, (and simply write $\mathcal{A}$ when $\alpha = 1$ or $r = 2$)

$$\mathcal{A}^\alpha_r F(x) := \left(\iint_{\Gamma^\alpha(x)} |F(y,t)|^r \, \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{r}}; \tag{2.1}$$

and for $r = \infty$,

$$\mathcal{A}^\alpha_\infty F(x) := \sup_{(y,t) \in \Gamma^\alpha(x)} |F(y,t)|,$$

again when $\alpha = 1$ we simply write $\mathcal{A}_\infty$. Then for all $0 < p, r < \infty$ the tent space T^p_r is defined as:

$$T^p_r := \{F \text{ measurable in } \mathbb{R}^{n+1}_+ : \mathcal{A}_r F \in L^p(\mathbb{R}^n)\}.$$

In the case $0 < p < \infty$ and $r = \infty$, we consider

$$T^p_\infty := \{F \in C(\mathbb{R}^{n+1}_+) : \mathcal{A}_\infty F \in L^p(\mathbb{R}^n)\}.$$

When $0 < r < \infty$ we identify functions which coincide almost everywhere and for $r = \infty$ we consider the norm in T^p_r given by $\|F\|_{T^p_r} := \|\mathcal{A}_r F\|_{L^p(\mathbb{R}^n)}$. It is easy to see that this effectively defines a norm. Furthermore, for $1 \le p < \infty$ and $1 \le r \le \infty$ these spaces are Banach spaces. To see this note that for $1 \le p, r < \infty$ and every compact set $K \subset \mathbb{R}^{n+1}_+$, there exist some $x_K \in \mathbb{R}^n$ and $c_1,\ c_2,\ c_K > 0$ such that

$$K \subset \{(y,t) \in \mathbb{R}^{n+1}_+ : c_1 < t < c_2, y \in B(x_K, c_K)\}.$$

Then,

$$
\begin{aligned}
\left(\iint_K |F(y,t)|^r dy\,dt\right)^{\frac{1}{r}} &\leq \sup_{\|\phi\|_{L^{r'}(K)}\leq 1} \iint_K |F(y,t)||\phi(y,t)|dy\,dt \\
&\leq c_2 \sup_{\|\phi\|_{L^{r'}(K)}\leq 1} \iint_K |F(y,t)||\phi(y,t)| \int_{B(y,t)} dx \frac{dy\,dt}{t^{n+1}} \\
&\leq c_2 \sup_{\|\phi\|_{L^{r'}(K)}\leq 1} \int_{B(x_K, c_K+c_2)} \iint_K |F(y,t)||\phi(y,t)| \frac{dy\,dt}{t^{n+1}} dx \\
&\leq c_2 c_1^{-(n+1)/r'} \sup_{\|\phi\|_{L^{r'}(K)}\leq 1} \int_{B(x_K, c_K+c_2)} \left(\int_{c_1}^{c_2} \int_{B(x,t)\cap B(x_K,c_K)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{r}} dx\, \|\phi\|_{L^{r'}(K)} \\
&\leq c_2 c_1^{-(n+1)/r'} |B(x_K, c_K+c_2)|^{\frac{1}{p'}} \|\mathcal{A}_r F\|_{L^p(\mathbb{R}^n)}; \qquad (2.2)
\end{aligned}
$$

in the case $1 \leq p < \infty$ and $r = \infty$, note that for each $(y,t) \in \mathbb{R}^{n+1}_+$, we have for all $x \in B(y,t)$ that $(y,t) \in \Gamma(x)$. Then,

$$|F(y,t)| \leq \inf_{x\in B(y,t)} \sup_{(y,t)\in\Gamma(x)} |F(y,t)| = \inf_{x\in B(y,t)} \mathcal{A}_\infty F(x).$$

Hence, integrating in $B(y,t)$, we have

$$|F(y,t)| = |B(y,t)|^{-\frac{1}{p}} \left(\int_{B(y,t)} |F(y,t)|^p dx\right)^{\frac{1}{p}} \leq c_{n,p} t^{-\frac{n}{p}} \|\mathcal{A}_\infty F\|_{L^p(\mathbb{R}^n)}, \quad \forall\, (y,t) \in \mathbb{R}^{n+1}_+. \qquad (2.3)$$

Now consider a Cauchy sequence $\{F_m\}_{m\in\mathbb{N}}$ in T^p_r, by (2.2) we have for all $1 \leq p, r < \infty$ that, for every compact subset K of $\mathbb{R}^{n+1}_+$, $\{F_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^r(K)$. Thus, for each $M \in \mathbb{N}$ if we consider the compact subsets

$$K_M := \{(y,t) \in \mathbb{R}^{n+1}_+ : 1/M \leq t \leq M,\ y \in B(0,M)\},$$

we have that there exists F_M such that $F_m \mathbf{1}_{K_M} \to F_M$ for all $M \in \mathbb{N}$. Next, consider $F \in L^r_{loc}(\mathbb{R}^{n+1}_+)$ such that $F_{|K_M} = F_M$. We have that $F \in T^p_r$. To see this, first note that, for each $M \in \mathbb{N}$,

$$\|F\mathbf{1}_{K_M}\|_{T^p_r} \leq \|(F_M - F_m)\mathbf{1}_{K_M}\|_{T^p_r} + \|F_m \mathbf{1}_{K_M}\|_{T^p_r} \leq C_M \|F_m - F_M\|_{L^r(K_M)} + \|F_m\|_{T^p_r}.$$

Besides, for all $\varepsilon > 0$, since $\{F_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in T^p_r, in particular is bounded in T^p_r. On the other hand, since $F_m \mathbf{1}_{K_M} \to F_M$, for all $M \in \mathbb{N}$, we have that for all $\varepsilon > 0$ there exists m_1 such that, for all $m \geq m_1$, $\|F_M - F_m\|_{L^r(K_M)} \leq \varepsilon$. Then, from the above inequality, we conclude that, for all $m \geq m_1$

$$\|F\mathbf{1}_{K_M}\|_{T^p_r} \leq C_M \varepsilon + \|F_m\|_{T^p_r} \leq C_M \varepsilon + \sup_{m\in\mathbb{N}} \|F_m\|_{T^p_r} < \infty.$$

Since this holds for all $\varepsilon > 0$, we get

$$\|F\mathbf{1}_{K_M}\|_{T^p_r} \leq \sup_{m\in\mathbb{N}} \|F_m\|_{T^p_r} < \infty, \quad \forall\, M \in \mathbb{N},$$

which implies that $F \in T^p_r$. Finally, let us show that $F_m \to F$ in T^p_r. We use again the fact that $\{F_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in T^p_r: for all $\varepsilon > 0$ there exists m_0 such that $\|F_m - F_{m'}\|_{T^p_r} \leq \varepsilon$, for all $m, m' \geq m_0$. On the other hand, since $F_{m_0}, F \in T^p_r$, we have that

$$\lim_{M\to\infty} \|\mathbf{1}_{\mathbb{R}^n\setminus K_M}(F_{m_0} - F)\|_{T^p_r} = 0.$$

Then, there exists M_0 such that, for all $M \geq M_0$, $\|\mathbf{1}_{\mathbb{R}^n\setminus K_M}(F_{m_0} - F)\|_{T^p_r} \leq \varepsilon$. Therefore, for fixed $M \geq M_0$ and for all $m \geq m_0$, we have

$$\|F - F_m\|_{T^p_r} = \|\mathbf{1}_{K_M}(F - F_m)\|_{T^p_r} + \|\mathbf{1}_{\mathbb{R}^n\setminus K_M}(F - F_m)\|_{T^p_r}$$

$$\begin{aligned}&\le C_M\|F-F_m\|_{L^r(K_M)} + \|\mathbf{1}_{\mathbb{R}^n\setminus K_M}(F_{m_0}-F_m)\|_{T^p_r} + \|\mathbf{1}_{\mathbb{R}^n\setminus K_M}(F-F_{m_0})\|_{T^p_r}\\ &\le C_M\|F-F_m\|_{L^r(K_M)} + 2\varepsilon.\end{aligned}$$

Consequently,

$$\lim_{m\to\infty}\|F-F_m\|_{T^p_r} \le C_M \lim_{m\to\infty}\|F_M-F_m\|_{L^r(K_M)} + 2\varepsilon = 2\varepsilon.$$

Thus, $\{F_m\}_{m\in\mathbb{N}}$ converges to F in T^p_r.

In the case $r=\infty$, note that (2.3) implies that every Cauchy sequence $\{F_m\}_{m\in\mathbb{N}}$ in T^p_∞, converges uniformly to a function F in $\mathbb{R}^{n+1}_+$. It is easy to see that $F\in T^p_\infty$. Indeed, for all $\varepsilon>0$, we have that there exists some m_0 such that for all $m\ge m_0$, $|F(y,t)-F_m(y,t)|\le\varepsilon$. Thus, for all $M\in\mathbb{N}$

$$\|F\mathbf{1}_{K_M}\|_{T^p_\infty} \le C_M\varepsilon + \sup_{m\in\mathbb{N}}\|F_m\|_{T^p_\infty},$$

which implies that $F\in T^p_\infty$. Then,

$$\lim_{M\to\infty}\|\mathbf{1}_{\mathbb{R}^n\setminus K_M}(F-F_m)\|_{T^p_\infty} = 0.$$

Consequently, proceeding as in the case $r<\infty$ we conclude that $\{F_m\}_{m\in\mathbb{N}}$ converges to F in T^p_∞.

Remark 2.4. *By* (2.2) *we also have that the functions in $L^r(\mathbb{R}^{n+1}_+)$ with compact support are dense in T^p_r, for all* $0<p,r<\infty$.

For all $F\in T^p_r$ we consider $F_M := F\mathbf{1}_{K_M}\in L^r(K_M)$ and note that $\mathcal{A}F_{M_1}\le\mathcal{A}F_{M_2}$, for all $M_1\le M_2$. Then, by the monotone convergence theorem we conclude that $\{F_M\}_{M\in\mathbb{N}}$ converges to F in T^p_r.

Another important remark is that the definition of tent spaces does not depend on the aperture of the cone Γ^α used to define the operator $\mathcal{A}^\alpha_r$ (meaning that different angles give rise to equivalent norms). This first appears for $r=2$ in [32, Section 3, Proposition 4]; and later P. Auscher gave in [4] the sharp dependence on the angle of the constant in the equivalence of those norms. The result for a general $0<r<\infty$ is the following:

Theorem 2.5. *For all* $0<p,r<\infty$ *and* $\alpha,\beta>0$*, there holds*

$$c\underline{h}(r,p,\alpha/\beta)\|\mathcal{A}^\alpha_rF\|_{L^p(\mathbb{R}^n)} \le \|\mathcal{A}^\beta_rF\|_{L^p(\mathbb{R}^n)} \le c\overline{h}(r,p,\alpha/\beta)\|\mathcal{A}^\alpha_rF\|_{L^p(\mathbb{R}^n)},$$

where $\underline{h}(r,p,\alpha/\beta) := \min\{(\alpha/\beta)^{-n/r},(\alpha/\beta)^{-n/p}\}$ *and* $\overline{h}(r,p,\alpha/\beta) := \max\{(\alpha/\beta)^{-n/r},(\alpha/\beta)^{-n/p}\}$.

Remark 2.6. *The result is not true for $p=\infty$. See [32, Remark in Section 5] and [32, Remak (a) in Section 6] for an example. This is one reason why the spaces T^∞_r are not defined in the obvious way.*

For $1<r<\infty$, consider the functional

$$\widehat{C}_rF(x) := \sup_{B\ni x}\left(\iint_{\widehat{B}}|F(y,t)|^r\frac{dy\,dt}{t}\right)^{\frac{1}{r}}$$

where $\widehat{B} := \{(y,t)\in\mathbb{R}^{n+1}_+ : d(y,\mathbb{R}^n\setminus B)\ge t\}$ is called the tent over B. The space T^∞_r is defined as:

$$T^\infty_r := \{F \text{ measurable function } : \widehat{C}_rF\in L^\infty(\mathbb{R}^n)\},$$

with the norm given by $\|F\|_{T^\infty_r} := \|\widehat{C}_rF\|_{L^\infty(\mathbb{R}^n)}$.

In [32] the authors proved the following results regarding tent spaces and the operators $\mathcal{A}_r$ and $\widehat{C}_r$.

Theorem 2.7. (*i*) *For all $1\le p<\infty$ and $1<r<\infty$ the dual of T^p_r is $T^{p'}_{r'}$, where p' and r' are the conjugated exponents of p and r, respectively. Besides, the paring*

$$\iint_{\mathbb{R}^{n+1}_+}F(y,t)G(y,t)\frac{dy\,dt}{t}$$

realizes $T^{p'}_{r'}$ as equivalent with the dual of T^p_r.

(*ii*) *For all* $1 < r < \infty$, *there hold*

$$\|\mathcal{A}_r F\|_{L^p(\mathbb{R}^n)} \le c(p,r)\|\widehat{C}_r F\|_{L^p(\mathbb{R}^n)},\ \forall\, 0 < p < \infty; \quad \text{and} \quad \|\widehat{C}_r F\|_{L^p(\mathbb{R}^n)} \le c'(p,r)\|\mathcal{A}_r F\|_{L^p(\mathbb{R}^n)},\ \forall\, r < p.$$

(*iii*) *For* $0 < p \le 1$ *and* $1 < r < \infty$ *the space* T^p_r *has an atomic decomposition. An atom in* T^p_r *is a measurable function* $A(x,t)$ *such that there exists a ball* $B \subset \mathbb{R}^n$ *with* $\operatorname{supp}(A) \subset \widehat{B}$, *and*

$$\left(\iint_{\widehat{B}} |A(x,t)|^r \frac{dx\,dt}{t} \right)^{\frac{1}{r}} \le |B|^{\frac{1}{r}-\frac{1}{p}}. \tag{2.8}$$

This can be found in [32, Section 8, Proposition 5]. The result is the following: *Let* $F \in T^p_r$, $0 < p \le 1$ *and* $1 < r < \infty$. *Then* $F = \sum_{i=1}^{\infty} \lambda_i A_i$, *where* A_i *are* T^p_r *atoms,* $\lambda_i \in \mathbb{C}$, *and* $\left(\sum_{i=1}^{\infty} |\lambda_i|^p\right)^{\frac{1}{p}} \lesssim \|F\|_{T^p_r}$. *Conversely, any such sum converges in* T^p_r *and* $\|\sum_{i=1}^{\infty} \lambda_i A_i\|_{T^p_r} \lesssim \left(\sum_{i=1}^{\infty} |\lambda_i|^p\right)^{\frac{1}{p}}$.

(*iv*) *([32, Section 7, Theorem 4]) Suppose* $1 \le p_0 < p_1 \le \infty$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$. *Then*

$$[T^{p_0}, T^{p_1}]_\theta = T^p,$$

where we denote by $[\ ,\ ]_\theta$ *the complex interpolation method described in [26] (see Appendix B).*

Besides, for all $0 < p, r < \infty$ we define the weak tent space wT^p_r as the space of all measurable functions F such that $\mathcal{A}_r F \in L^{p,\infty}(\mathbb{R}^n)$ endowed with the norm $\|F\|_{wT^p_r} := \|\mathcal{A}_r F\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : \mathcal{A}_r F(x) > \lambda\}|^{\frac{1}{p}}$.

We also have that the definition of these spaces does not depend on the aperture of the cone Γ^α used in the definition of the operator $\mathcal{A}^\alpha_r$.

Theorem 2.9. *Let* $0 < p, r, \alpha, \beta < \infty$, *there holds*

$$\|\mathcal{A}^\alpha_r F\|_{L^{p,\infty}(\mathbb{R}^n)} \approx \|\mathcal{A}^\beta_r F\|_{L^{p,\infty}(\mathbb{R}^n)}.$$

Proof. Without loss of generality, assume that $\alpha < \beta$. Then, for all $x \in \mathbb{R}^n$ we have that $\mathcal{A}^\alpha_r F(x) \le \mathcal{A}^\beta_r F(x)$. Thus,

$$\|\mathcal{A}^\alpha_r F\|_{L^{p,\infty}(\mathbb{R}^n)} \le \|\mathcal{A}^\beta_r F\|_{L^{p,\infty}(\mathbb{R}^n)}.$$

In order to see the converse inequality, we proceed as in the proof of [56, Lemma 6.2]. But first note that since $\mathcal{A}^\beta_r F = \alpha^n \mathcal{A}^{\frac{\beta}{\alpha}}_r \widetilde{F}$, where $\widetilde{F}(y,t) := F(y, t/\alpha)$, it suffices to prove that $\|\mathcal{A}^\beta_r F\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|\mathcal{A}_r F\|_{L^{p,\infty}(\mathbb{R}^n)}$, for all $\beta \ge 1$.

Fix $\lambda > 0$ and consider the sets $O_\lambda := \{x \in \mathbb{R}^n : \mathcal{A}_r F(x) > \lambda\}$, $E_\lambda := \mathbb{R}^n \backslash O_\lambda$, and, for $\gamma := 1 - \frac{1}{(6\beta)^n}$, the set of γ density $E^*_\lambda := \left\{x \in \mathbb{R}^n : \forall r > 0\ \frac{|E_\lambda \cap B(x,r)|}{|B(x,r)|} \ge \gamma\right\}$. Note that $O^*_\lambda := \mathbb{R}^n \setminus E^*_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_\lambda})(x) > 1/(6\beta)^n\}$.

We claim that for all $0 < r < \infty$,

$$\mathcal{A}^\beta_r F(x) \le (2\beta)^{\frac{n}{r}} \lambda, \text{ for all } x \in E^*_\lambda. \tag{2.10}$$

Since this is for all $\lambda > 0$, assuming it and applying that $\mathcal{M} : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$, we would have, for all $0 < p < \infty$,

$$\begin{aligned}\|\mathcal{A}^\beta_r F\|_{L^{p,\infty}(\mathbb{R}^n)} &= \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : \mathcal{A}^\beta_r F(x) > \lambda\}|^{1/p} \\ &= \sup_{\lambda>0} (2\beta)^{n/r} \lambda |\{x \in \mathbb{R}^n : \mathcal{A}^\beta_r f(x) > (2\beta)^{n/r}\lambda\}|^{1/p} \le (2\beta)^{n/r} \sup_{\lambda>0} \lambda |O^*_\lambda|^{1/p}\end{aligned}$$

$$\lesssim \beta^{n+n/r} \sup_{\lambda>0} \lambda |O_\lambda|^{1/p} = \beta^{n+n/r} \|\mathcal{A}_r F\|_{L^{p,\infty}(\mathbb{R}^n)}.$$

So it just remains to show (2.10). First, note that if $x \in E_\lambda^*$ then, for every $(y,t) \in \Gamma^{2\beta}(x)$, $B(y,t/2) \cap E_\lambda \neq \emptyset$. To prove this, suppose we had that $B(y,t/2) \subset O_\lambda$. Then, since $B(y,t/2) \subset B(x, 5\beta t/2)$,

$$\mathcal{M}(\mathbf{1}_{O_\lambda})(x) \geq \frac{|B(y,t/2)|}{|B(x,5\beta t/2)|} = \frac{1}{(5\beta)^n} > \frac{1}{(6\beta)^n},$$

which implies that $x \in O_\lambda^*$, a contradiction. Hence, there exists $y_0 \in B(y,t/2)$ (in particular $B(y,t/2) \subset B(y_0,t)$) such that $\mathcal{A}_r F(y_0) \leq \lambda$. Therefore, for all $(y,t) \in \Gamma^{2\beta}(x)$, with $x \in E_\lambda^*$,

$$\int_0^\infty \int_{B(y,t/2)} |F(x,t)|^2 \frac{dx\,dt}{t^{n+1}} \leq \int_0^\infty \int_{B(y_0,t)} |F(x,t)|^2 \frac{dx\,dt}{t^{n+1}} = \mathcal{A}_r F(y_0)^2 \leq \lambda^2. \tag{2.11}$$

On the other hand, for all $x \in \mathbb{R}^n$, we have that $B(x,\beta t) \subset \bigcup_i B(x_i, t/2)$, where $\{B(x_i,t)\}_i$ is a collection of $(2\beta)^n$ balls such that $x_i \in B(x, 2\beta t)$, or equivalently $(x_i,t) \in \Gamma^{2\beta}(x)$.

Therefore, we conclude that, for all $x \in E_\lambda^*$

$$\mathcal{A}_r^\beta(x) = \left(\int_0^\infty \int_{B(x,\beta t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{r}} \leq \left(\sum_i \int_0^\infty \int_{B(x_i,t/2)} |F(y,t)|^r \frac{d\,dt}{t^{n+1}} \right)^{\frac{1}{r}} \leq (2\beta)^{n/r}\lambda,$$

where we have used (2.11), since $x \in E_\lambda^*$ and $(x_i,t) \in \Gamma^{2\beta}(x)$. □

Remark 2.12. *Note that Theorem 2.9 is also true if we consider $w \in A_\infty$ and replace the space $L^{p,\infty}(\mathbb{R}^n)$ with $L^{p,\infty}(w)$ (which is defined in the obvious way: changing the Lebesgue measure by the measure given by the weight w). Besides, it can be also obtained as a consequence of [36, Theorem 2.1] with $\mathbb{X} = L^{p,\infty}(\mathbb{R}^n)$.*

Finally, for $0 < p \leq 1$, we define the space $\mathfrak{T}_r^p$ as the subspace of $F \in T_r^p$ having an atomic decomposition $\sum_{i=1}^\infty \lambda_i A_i$ such that the atoms A_i also satisfy $\int_{\mathbb{R}^n} A_i(x,t)dx = 0$ for a.e. $t > 0$ and $\forall i \geq 1$, and that $\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{\frac{1}{p}} < \infty$. This spaces will appear when we study the action of Calderón-Zygmund operator on tent spaces.

2.2 Action of operators on tent spaces

Tent spaces also appear if one wants to study maximal regularity operators arising from some linear or nonlinear partial differential equations ([64], [8]). In particular, one wants to understand how some (sub)linear operators act on them. More precisely the following two types of operators appear. First,

$$\mathcal{T}(F)(x,t) := T_t(F(\cdot,t))(x),$$

where T_t acts on functions on $\mathbb{R}^n$. Second,

$$\mathcal{T}(F)(x,t) := \int_0^\infty T_{t,s}(F(\cdot,s))(x)\frac{ds}{s},$$

where $T_{t,s}$ acts on functions on $\mathbb{R}^n$. For the second type, we refer to [12], [59], [8]. Positive results on T_2^p all rely on the use of $L^2(\mathbb{R}^n)$ off-diagonal estimates (or improved $L^{\min(p,2)}(\mathbb{R}^n) - L^{\max(p,2)}(\mathbb{R}^n)$ off diagonal estimates) and change of angle in the tent space norms (recall Theorem 2.5).

For the first type, there is a simple sufficient condition that also depends on the change of angle. Let us assume that T_t acts on $L^2(\mathbb{R}^n)$ functions with compact support and

$$\int_{B(x,t)} |T_t(f)(y)|^2\,dy \lesssim 2^{-2j\gamma} \int_{C_j(B(x,t))} |f(y)|^2\,dy \tag{2.13}$$

with some $\gamma \geq 0$, provided f is supported in $C_j(B(x,t))$, $j \geq 1$. Then

$$\mathcal{A}(\mathcal{T}(F))(x) \lesssim \sum_{j\geq 1} 2^{-j\gamma}\mathcal{A}^{2^{j+1}}(F)(x).$$

Using Theorem 2.5, we can conclude the T^p boundedness of $\mathcal{T}$ if $\gamma > n \max\left\{\frac{1}{2}, \frac{1}{p}\right\}$. Note in particular that if $\gamma \leq n/2$, this argument gives no boundedness, even for $p = 2$. Often, the operators T_t are assumed to be uniformly bounded on $L^2(\mathbb{R}^n)$, which gives T^2 boundedness of $\mathcal{T}$, whatever γ. Still, a condition $\gamma > 0$ does not seem to guarantee boundedness on T^p for a range of p about 2 in general. Thus, there is no available general criterion when $\gamma \leq n/2$.

If we let $T_t = T$ be (independent of t) a Calderón-Zygmund operator, then one obtains (2.13) with $\gamma = n/2$. Similarly we get $\gamma = n/2$ if we let $T_t = \mathcal{M}$ be the centered Hardy-Littlewood maximal operator: for locally integrable f, the maximal operator is defined by

$$\mathcal{M}(f)(x) := \sup_{\tau>0} \fint_{B(x,\tau)} |f(y)|\,dy. \tag{2.14}$$

As said, this argument does not apply.

On the other hand, it is well-known that if we replace $\mathcal{A}$ by the vertical norm $\mathcal{V}_2$ where

$$\mathcal{V}_r(F)(x) = \left(\int_0^\infty |F(x,t)|^r \frac{dt}{t}\right)^{\frac{1}{r}}, \quad x \in \mathbb{R}^n,$$

then,

$$\|\mathcal{V}_r(\mathcal{M}(F))\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{V}_r(F)\|_{L^p(\mathbb{R}^n)}, \tag{2.15}$$

which is the vector-valued maximal inequality of Fefferman-Stein, valid when $1 < p, r < \infty$ ([41]). It is thus a natural question whether $\mathcal{V}_r$ can be replaced by $\mathcal{A}_r$, that is whether the maximal operator, identified with its tensor product with the identity on functions of the t variable, is bounded on T^p_r.

A modern simple proof of (2.15) is by invoking extrapolation (see [35]): it suffices to prove

$$\|\mathcal{V}_r(\mathcal{M}(F))\|_{L^r(w)} \lesssim \|\mathcal{V}_r(F)\|_{L^r(w)}$$

for any $w \in A_r$ to obtain (2.15), and the latter follows from Muckenhoupt's theorem. Thus we are tempted to follow the same route and indeed, we shall prove

$$\|\mathcal{A}_r(\mathcal{M}(F))\|_{L^r(w)} \lesssim \|\mathcal{A}_r(F)\|_{L^r(w)}$$

for any $w \in A_r$ using simple upper bounds and known results. We note that the functionals $\mathcal{V}_r$ and $\mathcal{A}_r$ are not comparable on $L^p(\mathbb{R}^n)$ when $p \neq r$, as shown in [7]. Hence, one cannot deduce such results directly.

For other operators, we shall also show how extrapolation allows us to conclude tent space boundedness: we will consider Calderón-Zygmund operators, Riesz potentials and fractional maximal functions, in which case, one looks for T^p_r to T^q_r boundedness for some $q > p$. We will also consider singular non-integral operators such as the Riesz transform of elliptic operators to test applicability of our methods. In this case, it is a representation of the operator in the form $\int_0^\infty \theta_s \frac{ds}{s}$ that is essential and that replaces the representation by the kernel. We obtain tent space boundedness with limited range in p and r that is consistent with that of the $L^p(\mathbb{R}^n)$ theory.

For Calderón-Zygmund operators, we shall explore what happens when $p \leq 1$. At $p = 1$, we prove a weak-type inequality. We can also take advantage of cancellations in using atomic decompositions at the level of tent spaces. Then, atoms need to satisfy the additional condition

$$\int_{\mathbb{R}^n} A(x,t)\,dx = 0, \quad \text{for a.e. } t > 0$$

and we get results for $p > \frac{n}{n+1}$. Imposing more vanishing moments against polynomials allows us to get smaller values of p as it is the case with Hardy spaces on $\mathbb{R}^n$.

As easy corollaries, we obtain results in amalgam spaces in Appendix A.

As mentioned, if $(T_t)_{t>0}$ is a family of operators on $\mathbb{R}^n$ acting on (some) measurable functions, we let $\mathcal{T}$ be defined by

$$\mathcal{T}(F)(x,t) = T_t(F(\cdot,t))(x),$$

provided the formula makes sense, that is, provided $F(\cdot,t)$ belongs to an appropriate domain of T_t. If T is a single operator and $T_t = T$ for each $t > 0$ then $\mathcal{T} = T \otimes I$. In that case and from now on, we use the same notation by a slight abuse.

2.2.1 Hardy-Littlewood maximal operator

Theorem 2.16. *Let $\mathcal{M}$ be the centered Hardy-Littlewood maximal operator. For all $1 < r < \infty$,*
(*a*) $\mathcal{M} : T^p_r \to T^p_r$, *for all* $1 < p < \infty$;

(*b*) $\mathcal{M} : T^1_r \to wT^1_r$.

From this Theorem, we have some interesting corollaries:

Corollary 2.17. *Assume $(T_t)_{t>0}$ is a family of operators with $\sup_{t>0} |T_t(f)| \leq (\mathcal{M}|f|^\rho)^{1/\rho}$ for some $\rho \geq 1$. For all $\rho < p, r < \infty$,*

$$\mathcal{T} : T^p_r \to T^p_r. \tag{2.18}$$

This applies to the heat semigroup $e^{t^2\Delta}$ or the Poisson semigroup $e^{-t\sqrt{-\Delta}}$. Note that, in both cases, there is enough decay. Often, the sup norm is too strong a hypothesis. Here is a weaker one, applying for example to semigroups e^{-t^2L} associated to elliptic operators such as the ones in Section 2.2.4.

Corollary 2.19. *Assume $(T_t)_{t>0}$ is a family of operators with a kind of reverse Hölder estimate*

$$\left(\fint_{B(x,t)} |T_t(f)(y)|^s dy\right)^{\frac{1}{s}} \leq \left(\fint_{B(x,\alpha t)} |\mathcal{M}(|f|^\rho)(y)|\, dy\right)^{\frac{1}{\rho}}$$

for some $\alpha > 1$ and some $1 \leq \rho < s$, uniformly for all $(x,t) \in \mathbb{R}^{n+1}_+$. Then, for all (r,p) with $\rho < r \leq s$ and $\rho < p < \infty$,

$$\mathcal{T} : T^p_r \to T^p_r. \tag{2.20}$$

This follows from the pointwise inequality

$$\mathcal{A}_r(\mathcal{T}(F))(x) \leq \left(\mathcal{A}^{(\alpha)}_{\frac{r}{\rho}}(\mathcal{M}(|F|^\rho))(x)\right)^{\frac{1}{\rho}}$$

with $\mathcal{T}(F)(x,t) = T_t(F(\cdot,t))(x)$, and from Theorem 2.16.

Finally, Theorem 2.16 follows from the following pointwise inequality.

Lemma 2.21. *For all $x \in \mathbb{R}^n$, $t > 0$, and $1 < r < \infty$, and for all f locally r integrable, we have that*

$$\left(\fint_{B(x,t)} |\mathcal{M}(f)(y)|^r\, dy\right)^{\frac{1}{r}} \lesssim \left(\fint_{B(x,2t)} |f(y)|^r\, dy\right)^{\frac{1}{r}} + \mathcal{M}_u\left(\fint_{B(\cdot,t)} |f(z)| dz\right)(x), \tag{2.22}$$

where $\mathcal{M}_u$ is the uncentred maximal operator.

Proof. Fix $x \in \mathbb{R}^n$ and $t > 0$, and split the supremum into $0 < \tau \le t$ and $t < \tau$. Then,

$$\left(\fint_{B(x,t)} |\mathcal{M}(f)(y)|^r \, dy\right)^{\frac{1}{r}} \le \left(\fint_{B(x,t)} \left(\sup_{0<\tau\le t} \fint_{B(y,\tau)} |f(z)| dz\right)^r dy\right)^{\frac{1}{r}}$$

$$+ \left(\fint_{B(x,t)} \left(\sup_{\tau>t} \fint_{B(y,\tau)} |f(z)| dz\right)^r dy\right)^{\frac{1}{r}} =: I + II.$$

Now, since, for $0 < \tau \le t$ and $y \in B(x,t)$ it happens that $B(y,\tau) \subset B(x,2t)$,

$$I \le \left(\fint_{B(x,t)} \left(\sup_{0<\tau\le t} \fint_{B(y,\tau)} |f(z)\mathbf{1}_{B(x,2t)}(z)| dz\right)^r dy\right)^{\frac{1}{r}}$$

$$\le \left(\fint_{B(x,t)} |\mathcal{M}(f\mathbf{1}_{B(x,2t)})(y)|^r \, dy\right)^{\frac{1}{r}} \lesssim \left(\fint_{B(x,2t)} |f(y)|^r \, dy\right)^{\frac{1}{r}},$$

where in the last inequality we have used that $\mathcal{M} : L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ ([37, Theorem 2.5]).

As for II, note that, for $\xi, z \in \mathbb{R}^n$, $\xi \in B(z,t) \Leftrightarrow z \in B(\xi,t)$, and also that if $z \in B(y,\tau)$, $\xi \in B(z,t)$, and $\tau > t$, then $\xi \in B(y,2\tau)$. Besides, observe that the fact that $x \in B(y,t)$ and $\tau > t$ implies that $x \in B(y,2\tau)$. Hence, applying Fubini's theorem,

$$II = \left(\fint_{B(x,t)} \left(\sup_{\tau>t} \fint_{B(y,\tau)} |f(z)| \fint_{B(z,t)} d\xi \, dz\right)^r dy\right)^{\frac{1}{r}}$$

$$\le \left(\fint_{B(x,t)} \left(\sup_{\tau>t} \fint_{B(y,2\tau)} \fint_{B(\xi,t)} |f(z)| dz \, d\xi\right)^r dy\right)^{\frac{1}{r}} \lesssim \mathcal{M}_u \left(\fint_{B(\cdot,t)} |f(z)| dz\right)(x).$$

Gathering the estimates obtained for I and II gives us (2.22). □

Proof of Theorem 2.16, part (a).

We shall prove that, for all $w \in A_r$ and for all $F \in T_r^r$ (hence $F(\cdot,t)$ is locally r integrable for almost every $t > 0$), there holds

$$\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(F))(x)|^r w(x) dx \le C \int_{\mathbb{R}^n} |\mathcal{A}_r^2(F)(x)|^r w(x) dx. \tag{2.23}$$

From this, by Theorem 1.46, part (a), we have that, for all $1 < p < \infty$, $F \in T_r^r$ and $w_0 \in A_p$,

$$\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(F))(x)|^p w_0(x) dx \le C \int_{\mathbb{R}^n} |\mathcal{A}_r^2(F)(x)|^p w_0(x) dx.$$

In particular, for $w_0 \equiv 1$, we have that $w_0 \in A_p$ for all $1 < p < \infty$, then, applying Theorem 2.5, for all $F \in T_r^r$,

$$\|\mathcal{M}F\|_{T_r^p} = \left(\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(F))(x)|^p dx\right)^{\frac{1}{p}} \le C \left(\int_{\mathbb{R}^n} |\mathcal{A}_r^2(F)(x)|^p dx\right)^{\frac{1}{p}} \le C\|F\|_{T_r^p}. \tag{2.24}$$

Approximating T_r^p functions by compactly supported T_r^r functions, we conclude that (2.24) holds for functions $F \in T_r^p$ by the monotone convergence theorem.

Therefore, to finish the proof it just remains to show (2.23). This follows by (2.22) applied to $f = F(\cdot,t)$ and the fact that $\mathcal{M}_u : L^r(w) \to L^r(w)$, for all $w \in A_r$ ([70]). Then, for all $w \in A_r$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(F))(x)|^r w(x)dx &= \int_{\mathbb{R}^n}\int_0^\infty \fint_{B(x,t)} |\mathcal{M}(F(\cdot,t))(y)|^r \frac{dy\,dt}{t}\, w(x)dx \\
&\lesssim \int_{\mathbb{R}^n}\int_0^\infty \fint_{B(x,2t)} |F(y,t)|^r \frac{dy\,dt}{t}\, w(x)dx + \int_{\mathbb{R}^n}\int_0^\infty \left|\mathcal{M}_u\left(\fint_{B(\cdot,t)} |F(y,t)|dy\right)(x)\right|^r \frac{dt}{t}\, w(x)dx \\
&\lesssim \int_{\mathbb{R}^n} |\mathcal{A}_r^2(F)(x)|^r\, w(x)dx + \int_0^\infty \int_{\mathbb{R}^n} \left|\mathcal{M}_u\left(\fint_{B(\cdot,t)} |F(y,t)|dy\right)(x)\right|^r w(x)dx\frac{dt}{t} \\
&\lesssim \int_{\mathbb{R}^n} |\mathcal{A}_r^2(F)(x)|^r\, w(x)dx + \int_0^\infty \int_{\mathbb{R}^n} \left(\fint_{B(x,t)} |F(y,t)|dy\right)^r w(x)dx\frac{dt}{t} \\
&\lesssim \int_{\mathbb{R}^n} |\mathcal{A}_r^2(F)(x)|^r\, w(x)dx.
\end{aligned}$$

□

Proof of Theorem 2.16, part (*b*).

By (2.22) and the change of angle in tent spaces, for all $\lambda > 0$, we have that

$$\lambda|\{x \in \mathbb{R}^n : \mathcal{A}_r(\mathcal{M}(F))(x) > \lambda\}| \lesssim \|F\|_{T_r^1} + \lambda \left|\left\{x \in \mathbb{R}^n : \mathcal{V}_r(\mathcal{M}_u(\widetilde{F}))(x) > \frac{\lambda}{2}\right\}\right|,$$

where $\widetilde{F}(x,t) := \fint_{B(x,t)} |F(z,t)|\,dz$. Then, applying the Fefferman-Stein vector-valued weak type $(1,1)$ inequality [41], we control the second term in the above sum by

$$C\int_{\mathbb{R}^n}\left(\int_0^\infty |\widetilde{F}(x,t)|^r \frac{dt}{t}\right)^{\frac{1}{r}} dx \lesssim \|F\|_{T_r^1},$$

for some constant $C > 0$. Therefore, taking the supremum over all $\lambda > 0$, we conclude that

$$\|\mathcal{M}F\|_{wT_r^1} \lesssim \|F\|_{T_r^1}.$$

□

2.2.2 Calderón-Zygmund operators

Theorem 2.25. *Let $\mathcal{T}$ be a Calderón-Zygmund operator on $\mathbb{R}^n$ of order $\delta \in (0,1]$. For all $1 < r < \infty$,*

(*a*) $\mathcal{T} : T_r^p \to T_r^p$, *for all* $1 < p < \infty$;

(*b*) $\mathcal{T} : T_r^1 \to wT_r^1$;

(*c*) $\mathcal{T} : \mathfrak{T}_r^p \to T_r^p$, *for all* $\frac{n}{n+\delta} < p \leq 1$;

(*d*) $\mathcal{T} : \mathfrak{T}_r^p \to \mathfrak{T}_r^p$, *for all* $\frac{n}{n+\delta} < p \leq 1$, *if* $\mathcal{T}^*(1) = 0$.

Recall that $\mathcal{T}$ is a Calderón-Zygmund operator of order $\delta \in (0,1]$ if $\mathcal{T}$ is bounded on $L^2(\mathbb{R}^n)$ and has a kernel representation

$$\mathcal{T}(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy,$$

for almost every x not in the support of $f \in L^2(\mathbb{R}^n)$, with the kernel, K, satisfying the standard conditions: for some $\delta > 0$,

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \text{ for } x \neq y; \tag{2.26}$$

$$|K(x,y)-K(x,z)| \leq C\frac{|y-z|^{\delta}}{|x-y|^{n+\delta}}, \text{ for } |x-y|>2|y-z|; \tag{2.27}$$

$$|K(x,y)-K(w,y)| \leq C\frac{|x-w|^{\delta}}{|x-y|^{n+\delta}}, \text{ for } |x-y|>2|x-w|. \tag{2.28}$$

Classically, $\mathcal{T}$ extends to a bounded operator on $L^r(\mathbb{R}^n)$ for $1<r<\infty$ (see for instance [37, Theorem 5.10]) and the kernel representation holds also when $f \in L^r(\mathbb{R}^n)$. The following lemma gives us a useful pointwise inequality for Calderón-Zygmund operators. In particular, we use it in the proof of Theorem 2.25.

Lemma 2.29. *Let $\mathcal{T}$ be a Calderón-Zygmund operator and $f \in L^r(\mathbb{R}^n)$. We have, for $1<r<\infty$, and for each $x \in \mathbb{R}^n$ and all $t>0$,*

$$\left(\fint_{B(x,t)} |\mathcal{T}(f)(y)|^r dy\right)^{\frac{1}{r}} \lesssim \left(\fint_{B(x,2t)} |f(y)|^r dy\right)^{\frac{1}{r}} + \mathcal{T}_*(f)(x) + \mathcal{M}(f)(x), \tag{2.30}$$

where $\mathcal{T}_*(f)(x) := \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} K(x,y)f(y)dy\right|$.

Proof. Fix $x \in \mathbb{R}^n$ and $t>0$, consider the ball $B(x,2t)$ and write $f = f\mathbf{1}_{B(x,2t)} + f\mathbf{1}_{\mathbb{R}^n \setminus B(x,2t)} =: f_{loc} + f_{glob}$. Then

$$\left(\fint_{B(x,t)} |\mathcal{T}(f)(y)|^r dy\right)^{\frac{1}{r}} \leq \left(\fint_{B(x,t)} |\mathcal{T}(f_{loc})(y)|^r dy\right)^{\frac{1}{r}} + \left(\fint_{B(x,t)} |\mathcal{T}(f_{glob})(y)|^r dy\right)^{\frac{1}{r}} =: I + II.$$

Since $\mathcal{T}: L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$,

$$I \lesssim \left(\fint_{B(x,2t)} |f(y)|^r dy\right)^{\frac{1}{r}}.$$

As for II, apply the fact that, for $y \in B(x,t)$, $\{z : |x-z|>2t\} \subset \{z : |x-z|>2|x-y|\}$ and (2.28). Then,

$$\begin{aligned}
II &= \left(\fint_{B(x,t)} \left|\int_{\mathbb{R}^n} K(y,z) f_{glob}(z)\, dz\right|^r dy\right)^{\frac{1}{r}} = \left(\fint_{B(x,t)} \left|\int_{|x-z|>2t} K(y,z) f(z)\, dz\right|^r dy\right)^{\frac{1}{r}} \\
&= \left(\fint_{B(x,t)} \left|\int_{|x-z|>2t} (K(y,z)-K(x,z)+K(x,z)) f(z)\, dz\right|^r dy\right)^{\frac{1}{r}} \\
&\leq \left(\fint_{B(x,t)} \left(\int_{|x-z|>2t} |K(y,z)-K(x,z)|\,|f(z)|\, dz\right)^r dy\right)^{\frac{1}{r}} \\
&\quad + \left(\fint_{B(x,t)} \left|\int_{|x-z|>2t} K(x,z) f(z)\, dz\right|^r dy\right)^{\frac{1}{r}} \\
&\lesssim \left(\fint_{B(x,t)} \left(\int_{|x-z|>2t} \frac{|x-y|^{\delta}}{|x-z|^{n+\delta}} |f(z)|\, dz\right)^r dy\right)^{\frac{1}{r}} + \left|\int_{|x-z|>2t} K(x,z) f(z)\, dz\right| \\
&\lesssim \left(\fint_{B(x,t)} \left(\sum_{k=0}^{\infty} \int_{2^{k+1}t<|x-z|\leq 2^{k+2}t} \frac{|x-y|^{\delta}}{|x-z|^{n+\delta}} |f(z)|\, dz\right)^r dy\right)^{\frac{1}{r}} + \left|\int_{|x-z|>2t} K(x,z) f(z)\, dz\right| \\
&\lesssim \sum_{k=0}^{\infty} \frac{1}{2^{k\delta}} \left(\fint_{B(x,t)} \left(\fint_{B(x,2^{k+2}t)} |f(z)|\, dz\right)^r dy\right)^{\frac{1}{r}} + \left|\int_{|x-z|>2t} K(x,z) f(z)\, dz\right| \\
&\lesssim \sum_{k=0}^{\infty} \frac{1}{2^{k\delta}} \fint_{B(x,2^{k+2}t)} |f(z)|\, dz + \left|\int_{|x-z|>2t} K(x,z) f(z)\, dz\right| \\
&\lesssim \mathcal{M}(f)(x) + \mathcal{T}_*(f)(x).
\end{aligned}$$

□

Proof of Theorem 2.25, part (a)**.**

As we said above we first use (2.30) to prove a weighted version of the case $p = r$ for $\mathcal{T}$. We recall that we use the same notation $\mathcal{T}$ for its extension to tent spaces.

We consider $F \in T^r_r$ so that for almost every $t > 0$, $F(\cdot, t) \in L^r(\mathbb{R}^n)$ and all calculations make sense. For a weight $w \in A_r \cap RH_\infty$, by (2.30), Fubini's theorem, the fact that $\mathcal{T}_*, \mathcal{M} : L^r(w) \to L^r(w)$ (see for instance [30], [37, Theorem 7.13]), and applying [7, Proof of Proposition 2.3],

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)} |\mathcal{T}(F(\cdot,t))(y)|^r \frac{dy\,dt}{t^{n+1}} w(x)\,dx\right)^{\frac{1}{r}} \\
&\qquad \lesssim \left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}} w(x)\,dx\right)^{\frac{1}{r}} \\
&\qquad\quad + \left(\int_{\mathbb{R}^n}\int_0^\infty |\mathcal{T}_*(F(\cdot,t))(x)|^r \frac{dt}{t} w(x)\,dx\right)^{\frac{1}{r}} + \left(\int_{\mathbb{R}^n}\int_0^\infty |\mathcal{M}(F(\cdot,t))(x)|^r \frac{dt}{t} w(x)\,dx\right)^{\frac{1}{r}} \\
&\qquad \lesssim \left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}} w(x)\,dx\right)^{\frac{1}{r}} + \left(\int_{\mathbb{R}^n}\int_0^\infty |F(x,t)|^r \frac{dt}{t} w(x)\,dx\right)^{\frac{1}{2}} \\
&\qquad \lesssim \left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}} w(x)\,dx\right)^{\frac{1}{r}}.
\end{aligned}
$$

Therefore, for all $w \in A_r \cap RH_\infty$ and $F \in T^r_r$,

$$
\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)} |\mathcal{T}(F(\cdot,t))(y)|^r \frac{dy\,dt}{t^{n+1}} w(x)\,dx \lesssim \int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}} w(x)\,dx. \tag{2.31}
$$

In particular for $w \equiv 1$ and F as above, applying Theorem 2.5, we have that

$$
\|\mathcal{T}(F)\|_{T^r_r} \lesssim \|F\|_{T^r_r},
$$

where the estimate does not depend on F. This proves the case $p = r$. Note now that in view of (2.31), we can apply Theorem 1.46, part (c), for $p_- = 1$ and $p_+ = r$. Then, we obtain that, for all $1 < p < r$ and $w_0 \in A_p \cap RH_{\left(\frac{r}{p}\right)'}$, and all $F \in T^r_r$,

$$
\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,t)} |\mathcal{T}(F(\cdot,t))(y)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{q}{r}} w_0(x)\,dx \lesssim \int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,2t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{q}{r}} w_0(x)\,dx.
$$

Hence, taking $w_0 \equiv 1$, we have in particular that $w_0 \in A_p \cap RH_{\left(\frac{r}{p}\right)'}$. Then, applying Theorem 2.5, for $1 < p < r$ and all $F \in T^r_r$,

$$
\|\mathcal{T}(F)\|_{T^p_r} \lesssim \|F\|_{T^p_r}.
$$

We conclude by density of $T^r_r \cap T^p_r$ into T^p_r.

In order to prove the boundedness for $1 < r < p < \infty$, we use a duality argument. Take $F \in T^p_r \cap T^r_r$ and $G \in T^{p'}_{r'} \cap T^{r'}_{r'}$. By the previous argument and dualization (see Theorem 2.7, part (i)) we obtain,

$$
\int_{\mathbb{R}^n}\int_0^\infty |F(y,t)\mathcal{T}^*(G(\cdot,t))(y)| \frac{dt\,dy}{t} \lesssim \|F\|_{T^p_r}\|G\|_{T^{p'}_{r'}},
$$

where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$. Also

$$
\int_{\mathbb{R}^n}\int_0^\infty |\mathcal{T}(F(\cdot,t))(x)G(x,t)| \,\frac{dt\,dx}{t} \lesssim \|F\|_{T^r_r}\|G\|_{T^{r'}_{r'}} < \infty
$$

Thus, Fubini's theorem and

$$\int_{\mathbb{R}^n} \mathcal{T}(F(\cdot,t))(x)G(x,t)\,dx = \int_{\mathbb{R}^n} F(y,t)\mathcal{T}^*(G(\cdot,t))(y)\,dy$$

yield

$$\left| \int_{\mathbb{R}^n} \int_0^\infty \mathcal{T}(F(\cdot,t))(x)G(x,t)\frac{dt\,dx}{t} \right| = \left| \int_{\mathbb{R}^n} \int_0^\infty F(y,t)\mathcal{T}^*(G(\cdot,t))(y)\frac{dt\,dy}{t} \right| \lesssim \|F\|_{T_r^p} \|G\|_{T_{r'}^{p'}}.$$

Finally, taking the supremum over all G as above, such that $\|G\|_{T_{r'}^{p'}} \leq 1$, we conclude that, for all $F \in T_r^p \cap T_r^r$, $\|\mathcal{T}(F)\|_{T_r^p} \lesssim \|F\|_{T_r^p}$. By density, this allows us to extend the action of $\mathcal{T}$ to all $F \in T_r^p$. □

Remark that

$$\int_{\mathbb{R}^n} \int_0^\infty |\mathcal{T}(F(\cdot,t))(x)G(x,t)|\,\frac{dt\,dx}{t} < \infty$$

for all $F \in T_r^p$ and all $G \in T_{r'}^{p'}$ when $p = r$. But when $p \neq r$, the argument does not allow us to conclude the convergence of this integral for arbitrary $F \in T_r^p$ and $G \in T_{r'}^{p'}$. Of course, this inequality holds for the extension of $\mathcal{T}$ on T_r^p.

Proof of Theorem 2.25, part (b)**.**

Let $F \in T_r^r \cap T_r^1$, which is dense in T_r^1. It follows from (2.30) that

$$\mathcal{A}_r(\mathcal{T}(F)) \lesssim \mathcal{A}_r^2(F) + \mathcal{V}_r(\mathcal{M}(F)) + \mathcal{V}_r(\mathcal{T}_*(F)).$$

We need to estimate the $L^{1,\infty}(\mathbb{R}^n)$ norm of each term.

The first term has $L^1(\mathbb{R}^n)$ norm controlled by $c\|F\|_{T_r^1}$ for some constant $c > 0$ by change of angle in tent spaces (see Theorem 2.5).

For the second one, one applies Fefferman-Stein vector-valued weak type $(1,1)$ inequality, and next, the fact that the norm in $L^1(\mathbb{R}^n)$ of the vertical function $\mathcal{V}_r(F)$ is controlled by the norm in $L^1(\mathbb{R}^n)$ of the conical function $\mathcal{A}_r(F)$ (see [7]).

For the third term, the needed weak type estimate is

$$\left|\left\{x \in \mathbb{R}^n : \mathcal{V}_r(\mathcal{T}_*(F))(x) > \lambda\right\}\right| \lesssim \frac{\|\mathcal{V}_r(F)\|_{L^1(\mathbb{R}^n)}}{\lambda}.$$

This should be known but as we have not been able to locate a proof, we provide one for the reader's comfort. Once this is proved, we use again the result in [7] mentioned above.

Fix $\lambda > 0$ and consider the set

$$\Omega_\lambda := \{x \in \mathbb{R}^n : \mathcal{M}_u(\mathcal{V}_r(F))(x) > \lambda\},$$

where we recall that $\mathcal{M}_u$ represents the uncentred Hardy-Littlewood maximal operator. We have that Ω_λ is open and, since $\|\mathcal{V}_r(F)\|_{L^1(\mathbb{R}^n)} < \infty$, we conclude that $|\Omega_\lambda| < \infty$, and in particular $\mathbb{R}^n \setminus \Omega_\lambda \neq \emptyset$. Therefore, we can take a Whitney decomposition $\Omega_\lambda = \bigcup_{i\in\mathbb{N}} Q_i$, where Q_i are dyadic and disjoint cubes such that

$$\sqrt{n}\,\ell(Q_i) \leq \operatorname{dist}(Q_i, \mathbb{R}^n \setminus \Omega_\lambda) < 4\sqrt{n}\,\ell(Q_i).$$

Hence,

$$\mathcal{V}_r(F)(x) \leq \lambda, \ \text{ for a.e. } x \in \mathbb{R}^n \setminus \Omega_\lambda, \quad \fint_{Q_i} |\mathcal{V}_r(F)(x)|\,dx \leq 8^n\lambda, \quad \text{and} \quad |\Omega_\lambda| \leq \frac{1}{\lambda}\int_{\Omega_\lambda} |\mathcal{V}_r(F)(x)|\,dx.$$

Then if we set

$$\mathcal{G} = \mathcal{V}_r(F)\mathbf{1}_{\mathbb{R}^n\setminus\Omega_\lambda} + \sum_{i\in\mathbb{N}} \left(\fint_{Q_i} \mathcal{V}_r(F)\right)\mathbf{1}_{Q_i}, \ \text{ and } \mathcal{B} = \sum_{i\in\mathbb{N}} \left(\mathcal{V}_r(F) - \fint_{Q_i} \mathcal{V}_r(F)\right)\mathbf{1}_{Q_i}$$

we have that $\mathcal{V}_r(F) = \mathcal{G} + \mathcal{B}$ is a Calderón-Zygmund decomposition of $\mathcal{V}_r(F)$ at heigh λ satisfying:

$$|\mathcal{G}(x)| \leq 10^n\lambda, \text{ for a.e. } x \in \mathbb{R}^n, \quad \|\mathcal{G}\|_{L^r(\mathbb{R}^n)}^r \leq (10^n\lambda)^{r-1}\|\mathcal{V}_r(F)\|_{L^1(\mathbb{R}^n)},$$

$$\int_{Q_i} \mathcal{B}(x)dx = 0, \fint_{Q_i} |\mathcal{B}(x)|dx \leq 2\fint_{Q_i} |\mathcal{V}_r(F)(x)|dx, \text{ and } \|\mathcal{B}\|_{L^1(\mathbb{R}^n)} \leq 2\|\mathcal{V}_r(F)\|_{L^1(\mathbb{R}^n)}.$$

Now set $F = G + H$, where

$$G(x,t) = F(x,t)\mathbf{1}_{\mathbb{R}^n\setminus\Omega_\lambda}(x) + \sum_{i\in\mathbb{N}} \mathbf{1}_{Q_i}(x)\fint_{Q_i} F(y,t)\,dy,$$

and

$$H(x,t) = \sum_{i\in\mathbb{N}} \mathbf{1}_{Q_i}(x)\left(F(x,t) - \fint_{Q_i} F(y,t)\,dy\right) =: \sum_{i\in\mathbb{N}} H_i(x,t).$$

Then,

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : \mathcal{V}_r(\mathcal{T}_*(F))(x) > \lambda\right\}\right| \\
&\qquad \leq \left|\left\{x \in \mathbb{R}^n : \mathcal{V}_r(\mathcal{T}_*(G))(x) > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : \mathcal{V}_r(\mathcal{T}_*(H))(x) > \frac{\lambda}{2}\right\}\right| =: I + II.
\end{aligned}$$

Applying Chebychev's inequality and the $L^r(\mathbb{R}^n)$ boundedness of $\mathcal{T}_*$, we obtain

$$\begin{aligned}
I &\lesssim \frac{1}{\lambda^r}\int_{\mathbb{R}^n} \mathcal{V}_r(\mathcal{T}_*(G))(x)^r\,dx \\
&= \frac{1}{\lambda^r}\int_0^\infty\int_{\mathbb{R}^n} |\mathcal{T}_*(G(\cdot,t))(x)|^r\,dx\frac{dt}{t} \lesssim \frac{1}{\lambda^r}\int_0^\infty\int_{\mathbb{R}^n} |G(x,t)|^r\,dx\frac{dt}{t} \\
&= \frac{1}{\lambda^r}\int_{\mathbb{R}^n}\int_0^\infty \left|F(x,t)\mathbf{1}_{\mathbb{R}^n\setminus\Omega_\lambda}(x) + \sum_{i\in\mathbb{N}} \mathbf{1}_{Q_i}(x)\fint_{Q_i} |F(y,t)|\,dy\right|^r \frac{dt}{t}\,dx \\
&\lesssim \frac{1}{\lambda^r}\int_{\mathbb{R}^n} \left|\mathbf{1}_{\mathbb{R}^n\setminus\Omega_\lambda}(x)\mathcal{V}_r(F)(x) + \sum_{i\in\mathbb{N}} \mathbf{1}_{Q_i}(x)\fint_{Q_i} \mathcal{V}_r(F)(y)\,dy\right|^r\,dx \\
&= \frac{1}{\lambda^r}\|\mathcal{G}\|_{L^r(\mathbb{R}^n)}^r \lesssim \frac{1}{\lambda}\|\mathcal{V}_r(F)\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

As for the estimate of II, note that

$$II \lesssim \left|\bigcup_{i\in\mathbb{N}} 2\sqrt{n}\,Q_i\right| + \left|\left\{x \in \mathbb{R}^n \setminus \left(\bigcup_{i\in\mathbb{N}} 2\sqrt{n}\,Q_i\right) : \mathcal{V}_r(\mathcal{T}_*(H))(x) > \frac{\lambda}{2}\right\}\right|.$$

Then, since

$$\left|\bigcup_{i\in\mathbb{N}} 2\sqrt{n}\,Q_i\right| \lesssim |\Omega_\lambda| \lesssim \frac{1}{\lambda}\|\mathcal{V}_r(F)\|_{L^1(\mathbb{R}^n)},$$

we just need to consider the second term in the previous sum. For $t > 0$, and $x \in \mathbb{R}^n \setminus \left(\bigcup_{j\in\mathbb{N}} 2\sqrt{n}\,Q_j\right)$, let us study the $\mathcal{T}_*(H)(x,t)$. Pick $\varepsilon > 0$ and consider

$$\left|\int_{|x-y|>\varepsilon} K(x,y)H(y,t)\,dy\right| = \left|\sum_{i\in\mathbb{N}}\int_{|x-y|>\varepsilon} K(x,y)H_i(y,t)\,dy\right|.$$

We distinguish three possible cases in the series. Case 1: $Q_i \subset B(x,\varepsilon)$. Then, $Q_i \cap (\mathbb{R}^n \setminus B(x,\varepsilon)) = \emptyset$, and consequently

$$\left| \int_{|x-y|>\varepsilon} K(x,y) H_i(y,t)\, dy \right| = 0.$$

Case 2: $Q_i \subset \mathbb{R}^n \setminus B(x,\varepsilon)$. Call y_i the centre of Q_i and $\ell(Q_i)$ its length. Since $x \in \mathbb{R}^n \setminus 2\sqrt{n}\, Q_i$, we have $|x-y_i| > 2|y-y_i|$ for any $y \in Q_i$. As $\operatorname{supp}(H_i) \subset Q_i \subset \mathbb{R}^n \setminus B(x,\varepsilon)$, we can use the mean value $\int_{\mathbb{R}^n} H_i(y,t)\, dy = 0$ to obtain

$$\begin{aligned} \left| \int_{|x-y|>\varepsilon} K(x,y) H_i(y,t)\, dy \right| &\le \int_{Q_i} |K(x,y) - K(x,y_i)| |H_i(y,t)|\, dy \\ &\lesssim \frac{\ell(Q_i)^\delta}{|x-y_i|^{n+\delta}} \int_{Q_i} |H_i(y,t)|\, dy \lesssim \frac{\ell(Q_i)^\delta}{|x-y_i|^{n+\delta}} \int_{Q_i} |F(y,t)|\, dy. \end{aligned}$$

Case 3: $B(x,\varepsilon) \cap Q_i \neq \emptyset$ but $Q_i \not\subset B(x,\varepsilon)$. Note that $\varepsilon > \sqrt{n}\, \ell(Q_i)/2$. Indeed, if $x_0 \in B(x,\varepsilon) \cap Q_i$, $\sqrt{n}\, \ell(Q_i) \le |x-y_i| \le |x_0 - x| + |x_0 - y_i| < \varepsilon + \frac{\sqrt{n}\, \ell(Q_i)}{2}$ hence $\sqrt{n}\, \ell(Q_i)/2 < \varepsilon$. It follows that $Q_i \subset B(x, 3\varepsilon)$. Hence,

$$\begin{aligned} \left| \int_{|x-y|>\varepsilon} K(x,y) H_i(y,t)\, dy \right| &\le \int_{\varepsilon<|x-y|<3\varepsilon} \frac{1}{|x-y|^n} |H_i(y,t)|\, dy \\ &\lesssim \fint_{B(x,3\varepsilon)} |F(y,t)| \mathbf{1}_{Q_i}(y)\, dy + \fint_{B(x,3\varepsilon)} \fint_{Q_i} |F(z,t)|\, dz \mathbf{1}_{Q_i}(y)\, dy \lesssim \fint_{B(x,3\varepsilon)} |F(y,t)| \mathbf{1}_{Q_i}(y)\, dy. \end{aligned}$$

It follows that

$$\left| \int_{|x-y|>\varepsilon} K(x,y) H(y,t)\, dy \right| \lesssim \sum_{i\in\mathbb{N}} \frac{\ell(Q_i)^\delta}{|x-y_i|^{n+\delta}} \int_{Q_i} |F(y,t)|\, dy + \fint_{B(x,3\varepsilon)} |F(y,t)| \mathbf{1}_{\Omega_\lambda}(y)\, dy.$$

Taking the supremum over all $\varepsilon > 0$, we obtain,

$$\mathcal{T}_*(H)(x,t) \le \sum_{i\in\mathbb{N}} \frac{\ell(Q_i)^\delta}{|x-y_i|^{n+\delta}} \int_{Q_i} |F(y,t)|\, dy + \mathcal{M}(F)(x,t).$$

Therefore, by Minkowski's inequality

$$\mathcal{V}_r(\mathcal{T}_*(H))(x) \le \sum_{i\in\mathbb{N}} \frac{\ell(Q_i)^\delta}{|x-y_i|^{n+\delta}} \int_{Q_i} |\mathcal{V}_r(F)(y)|\, dy + \mathcal{V}_r(\mathcal{M}(F))(x).$$

Consequently, applying the Fefferman-Stein weak type $(1,1)$ inequality and Chebychev's inequality

$$\begin{aligned} &\left| \left\{ x \in \mathbb{R}^n \setminus \left(\bigcup_{i\in\mathbb{N}} 2\sqrt{n}\, Q_i \right) : \mathcal{V}_r(\mathcal{T}_*(H))(x) > \frac{\lambda}{2} \right\} \right| \\ &\quad \lesssim \left| \left\{ x \in \mathbb{R}^n : \mathcal{V}_r(\mathcal{M}(F))(x) > \frac{\lambda}{4} \right\} \right| + \frac{1}{\lambda} \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n \setminus 2\sqrt{n}\, Q_i} \frac{\ell(Q_i)^\delta}{|x-y_i|^{n+\delta}}\, dx \int_{Q_i} |\mathcal{V}_r(F)(y)|\, dy \\ &\quad \lesssim \frac{1}{\lambda} \left(\|\mathcal{V}_r(F)\|_{L^1(\mathbb{R}^n)} + \int_{\Omega_\lambda} |\mathcal{V}_r(F)(y)|\, dy \right) \lesssim \frac{1}{\lambda} \|\mathcal{V}_r F\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

□

Proof of Theorem 2.25, part (c)**.**

Recall that for $p \le 1$ and $1 < r < \infty$, every function in T^p_r has an atomic decomposition (see Theorem 2.7), part (iii)). Let us now introduce, for $0 < p \le 1$ and $1 < r < \infty$, a subspace of T^p_r that we denote by $\mathfrak{T}^p_r$. We say that A is a $\mathfrak{T}^p_r$ atom if it is a T^p_r atom and satisfies $\int_{\mathbb{R}^n} A(x,t)dx = 0$ for a.e. $t > 0$. This integral makes sense as

$$\left(\int_0^\infty \left(\int_{\mathbb{R}^n} |A(x,t)|\,dx\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} \le \left(\iint_{\widehat{B}} |A(x,t)|^r\,dx\frac{dt}{t}\right)^{\frac{1}{r}} |B|^{1-\frac{1}{r}} \le |B|^{1-\frac{1}{p}} < \infty.$$

We define $\mathfrak{T}^p_r$ as the subspace of $F \in T^p_r$ such that F has an atomic decomposition with A_i being a $\mathfrak{T}^p_r$ atom and $\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{\frac{1}{p}} < \infty$.

The reason to introduce those spaces is because, for $0 < p \le 1$, we can not obtain boundedness of singular integrals (and in general of Calderón-Zygmund operators) from the tent space T^p_r to itself. If we want to arrive into T^p_r, an option is to take functions in $\mathfrak{T}^p_r$. Note that T^p_r atoms, hence $\mathfrak{T}^p_r$ atoms, belong to T^r_r.

Lemma 2.32. *Suppose that $\mathcal{U} : T^r_r \to T^r_r$ is a linear and bounded operator and that there exists $C < \infty$ such that for all $\mathfrak{T}^p_r$ atoms A, $\|\mathcal{U}(A)\|_{T^p_r} \le C$. Then, $\mathcal{U}$ has a bounded extension from $\mathfrak{T}^p_r$ to T^p_r.*

Proof. Let A be a $\mathfrak{T}^p_r$ atom such that $\mathrm{supp}(A) \subset \widehat{B}$, for some ball $B \subset \mathbb{R}^n$. Defining, for $0 < \eta < \rho$, where ρ is the radius of B,

$$A_\eta(y,t) := \begin{cases} A(y,t) \text{ if } t > \eta, \\ 0 \text{ if } t \le \eta, \end{cases}$$

we have that $A - A_\eta$ are $\mathfrak{T}^q_r$ atoms, uniformly in η, thus

$$\|A - A_\eta\|_{\mathfrak{T}^p_r} \le |B|^{\frac{1}{p}-\frac{1}{r}} \left(\iint_{\widehat{B}} |A(x,t) - A_\eta(x,t)|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \to 0$$

by the dominated convergence theorem. This and the fact that finite linear combinations of $\mathfrak{T}^p_r$ atoms are dense in $\mathfrak{T}^p_r$ by definition, imply that the set E_r of compactly supported functions φ in $\mathbb{R}^{n+1}_+$ that are r-integrable and $\int_{\mathbb{R}^n} \varphi(x,t)\,dx = 0$ for a.e. $t > 0$ is dense in $\mathfrak{T}^p_r$. Then, let $F \in E_r$ and take a decomposition $F = \sum_{i=0}^\infty \lambda_i A_i$, where A_i are $\mathfrak{T}^p_r$ atoms and $\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{\frac{1}{q}} \le 2\|F\|_{\mathfrak{T}^p_r}$. Since the t support of F is contained in some interval $[a,b]$, we may eliminate the atoms associated to balls with radii less than a. Following the proof of Theorem 4.9 in [12], we obtain that the decomposition converges in T^r_r. Thus we may write

$$\mathcal{U}(F) = \sum_{i=0}^\infty \lambda_i\,\mathcal{U}(A_i)$$

and use the hypothesis to conclude that $\|\mathcal{U}(F)\|_{T^p_r} \le 2C\|F\|_{\mathfrak{T}^p_r}$. By density, we conclude the argument. □

We say that a function M is a T^p_r molecule if there exists a ball $B \subset \mathbb{R}^n$ such that, for some $\varepsilon > 0$,

$$\left(\iint_{\widehat{4B}} |M(x,t)|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \le |4B|^{\frac{1}{r}-\frac{1}{p}}$$

and, for all $j \ge 2$,

$$\left(\iint_{\widehat{C}_j} |M(x,t)|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \le 2^{-(j+1)\varepsilon}|2^{j+1}B|^{\frac{1}{r}-\frac{1}{p}},$$

where we define $\widehat{C_j} := \widehat{2^{j+1}B} \setminus \widehat{2^jB}$ and $\widehat{C_1} = \widehat{4B}$. By writing $M = \sum_{j\geq 1} \mathbf{1}_{\widehat{C}_j} M$ and observing that the functions $\mathbf{1}_{\widehat{C}_j} M$ are T^p_r atoms up to factor $2^{-(j+1)\varepsilon}$, we obtain $\|M\|_{T^p_r} \leq \left(\sum_{j\geq 1} |2^{-(j+1)\varepsilon}|^p\right)^{\frac{1}{p}}$.

Let us finally prove part (c) of Theorem 2.25. We follow the same scheme as in [33] and show that Calderón-Zygmund operators of order $\delta \in (0, 1]$ apply $\mathfrak{T}^p_r$ atoms to T^p_r molecules, provided that $p > \frac{n}{n+\delta}$, up to a constant that depends uniquely on δ, n, r, p and the properties of the operator. From the previous lemma, this suffices to conclude.

Let A be a $\mathfrak{T}^p_r$ atom. Let B be a ball such that $\operatorname{supp} A \subset \widehat{B}$, and

$$\left(\iint_{\widehat{B}} |A(x,t)|^r\right)^{\frac{1}{r}} \leq |B|^{\frac{1}{r}-\frac{1}{p}}.$$

We shall show that, for $\varepsilon = n + \delta - \frac{n}{p}$ (which is positive since $p > \frac{n}{n+\delta}$),

(1) $\left(\iint_{\widehat{4B}} |\mathcal{T}(A(\cdot,t))(x)|^r\right)^{\frac{1}{r}} \lesssim |4B|^{\frac{1}{r}-\frac{1}{p}}$;

(2) for $j \geq 2$, $\left(\iint_{\widehat{C}_j} |\mathcal{T}(A(\cdot,t))(x)|^r\right)^{\frac{1}{r}} \lesssim 2^{-(j+1)\varepsilon} |B_{j+1}|^{\frac{1}{r}-\frac{1}{p}}$.

For each $j \geq 2$, denote by $r_j := 2^j r_B$ and $B_j := B(x_B, r_j)$. Besides, recall that $\widehat{C_1} := \widehat{B_2}$ and $\widehat{C_j} := \widehat{B_{j+1}} \setminus \widehat{B}_j$, for all $j \geq 2$.

We start by proving (1). Since $\mathcal{T}$ is bounded on T^r_r, we have that

$$\left(\iint_{\widehat{4B}} |\mathcal{T}(A(\cdot,t))(x)|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \lesssim \left(\iint_{\widehat{B}} |A(x,t)|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \lesssim |4B|^{\frac{1}{r}-\frac{1}{p}}.$$

On the other hand, for $j \geq 2$, because $A(x,t) = 0$ for $t > r_B$, the radius of B,

$$\left(\iint_{\widehat{C}_j} |\mathcal{T}(A(\cdot,t))(x)|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \leq \left(\int_0^{r_B} \int_{B_{j+1}\setminus B_{j-1}} |\mathcal{T}(A(\cdot,t))(x)|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}}.$$

Now, applying the fact that $\int_{\mathbb{R}^n} A(x,t)\,dx = 0$ for a.e. $t > 0$, and the property (2.27) of the kernel K, we obtain that

$$\begin{aligned}
I &\leq \left(\int_0^{r_B} \int_{r_{j-1}\leq |x-x_B|<r_{j+1}} \left|\int_{\mathbb{R}^n} K(x,y)A(y,t)\,dy\right|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \\
&= \left(\int_0^{r_B} \int_{r_{j-1}\leq |x-x_B|<r_{j+1}} \left|\int_{\mathbb{R}^n} (K(x,y)-K(x,x_B))A(y,t)\,dy\right|^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \\
&\lesssim \left(\int_0^{r_B} \int_{r_{j-1}\leq |x-x_B|<r_{j+1}} \left(\int_{\mathbb{R}^n} \frac{|x_B-y|^\delta}{|x-x_B|^{n+\delta}} |A(y,t)|\,dy\right)^r \frac{dx\,dt}{t}\right)^{\frac{1}{r}} \\
&\lesssim \left(\int_0^{r_B} \int_{r_{j-1}\leq |x-x_B|<r_{j+1}} \fint_B |A(y,t)|^r\,dy \frac{dx\,dt}{t}\right)^{\frac{1}{r}} 2^{-(j+1)(n+\delta)} \\
&\lesssim \left(\int_0^{r_B} \int_B |A(y,t)|^r \frac{dy\,dt}{t}\right)^{\frac{1}{r}} 2^{-(j+1)\left(n\left(1-\frac{1}{r}\right)+\delta\right)} \\
&\lesssim 2^{-(j+1)\left(n\left(1-\frac{1}{r}\right)+\delta\right)} |B|^{\frac{1}{r}-\frac{1}{p}} = 2^{-(j+1)\left(n+\delta-\frac{n}{p}\right)} |2^{j+1}B|^{\frac{1}{r}-\frac{1}{p}}.
\end{aligned}$$

This shows (2). ◻

Proof of Theorem 2.25, part (d)**.**

Remark that if M is a T^p_r molecule, then

$$\left(\int_0^\infty \left(\int_{\mathbb{R}^n} |M(x,t))|\,dx\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} \lesssim \sum_{j\geq 1}\left(\int_0^\infty \left(\int_{\mathbb{R}^n} \mathbf{1}_{\widehat{C}_j}(x,t)|M(x,t)|\,dx\right)^r \frac{dt}{t}\right)^{\frac{1}{r}}$$
$$\lesssim \sum_{j\geq 1}\left(\iint_{\widehat{C}_j} |M(x,t)|^r\,dx\frac{dt}{t}\right)^{\frac{1}{r}} |B_{j+1}|^{1-\frac{1}{r}} \lesssim \sum_{j\geq 1} 2^{-(j+1)\varepsilon}|B_{j+1}|^{1-\frac{1}{p}} \lesssim |B|^{1-\frac{1}{p}} < \infty$$

as $1-\frac{1}{p}\leq 0$. Therefore, if, in addition, $\int_{\mathbb{R}^n} M(x,t)\,dx = 0$, for a.e. $t>0$, we say that M is a $\mathfrak{T}^p_r$ molecule. A $\mathfrak{T}^p_r$ molecule can be written as a series of $\mathfrak{T}^p_r$ atoms, as we see in the next result.

Proposition 2.33. *There exists a constant $C<\infty$ such that given a $\mathfrak{T}^p_r$ molecule M, we have that $M\in\mathfrak{T}^p_r$, with $\|M\|_{\mathfrak{T}^p_r}\leq C$.*

Proof. Let M be a $\mathfrak{T}^p_r$ molecule with associated ball $B = B(x_B, r_B)$. Following the notation in the previous proof, write

$$M(x,t) = \sum_{j=1}^\infty \left(M(x,t)\mathbf{1}_{\widehat{C}_j}(x,t) - \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{C}_j}(y,t)M(y,t)\,dy\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|}\right) + \sum_{j=1}^\infty \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{C}_j}(y,t)M(y,t)\,dy\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|}.$$

For every $j\geq 1$, we define

$$\alpha_j(x,t) := M(x,t)\mathbf{1}_{\widehat{C}_j}(x,t) - \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{C}_j}(y,t)M(y,t)\,dy\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|},$$

and observe that $\operatorname{supp}\alpha_j \subset B_{j+1}\times(0,r_{j+1}]\subset \widehat{B_{j+2}}$ and

$$\int_{\mathbb{R}^n}\alpha_j(x,t)\,dx = \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{C}_j}(y,t)M(y,t)\,dy\left(1-\int_{\mathbb{R}^n}\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|}\,dx\right) = 0.$$

Besides,

$$\left(\iint_{\widehat{B_{j+2}}} |\alpha_j(x,t)|^r\frac{dx\,dt}{t}\right)^{\frac{1}{r}} \leq \left(\iint_{\widehat{C}_j} |M(x,t)|^r\frac{dx\,dt}{t}\right)^{\frac{1}{r}}$$
$$+\left(\iint_{\widehat{B_{j+2}}} \mathbf{1}_{B_{j+1}}(x)\left(\frac{1}{|B_{j+1}|}\int_{B_{j+1}}\mathbf{1}_{\widehat{C}_j}(y,t)|M(y,t)|\,dy\right)^r\frac{dx\,dt}{t}\right)^{\frac{1}{r}} \leq 2\left(\iint_{\widehat{C}_j}|M(x,t)|^r\frac{dx\,dt}{t}\right)^{\frac{1}{r}}$$
$$\leq 2^{-(j+1)\varepsilon+1}|B_{j+1}|^{\frac{1}{r}-\frac{1}{p}} = c2^{-j\varepsilon}|B_{j+2}|^{\frac{1}{r}-\frac{1}{p}},$$

where c depends on ε, r, p only. Therefore, $A_j := \frac{2^{j\varepsilon}}{c}\alpha_j$ is a $\mathfrak{T}^p_r$ atom, for all $j\geq 1$.

On the other hand, note that

$$\sum_{j=1}^\infty \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{C}_j}(y,t)M(y,t)\,dy\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|} = \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{B_2}}(y,t)M(y,t)\,dy\frac{\mathbf{1}_{B_2}(x)}{|B_2|}$$
$$+\sum_{j=2}^\infty\left(\int_{\mathbb{R}^n}\mathbf{1}_{\widehat{B_{j+1}}}(y,t)M(y,t)\,dy - \int_{\mathbb{R}^n}\mathbf{1}_{\widehat{B_j}}(y,t)M(y,t)\,dy\right)\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|}$$

$$= \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{B_{j+1}}}(y,t) M(y,t)\, dy \left(\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|} - \frac{\mathbf{1}_{B_{j+2}}(x)}{|B_{j+2}|} \right).$$

Then, considering

$$\beta_j(x,t) := \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{B_{j+1}}}(y,t) M(y,t)\, dy \left(\frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|} - \frac{\mathbf{1}_{B_{j+2}}(x)}{|B_{j+2}|} \right),$$

we have that $\operatorname{supp}\beta_j \subset \widehat{B_{j+3}}$, and that

$$\int_{\mathbb{R}^n} \beta_j(x,t) dx = 0.$$

Besides, since, for a. e. $t > 0$,

$$\int_{\mathbb{R}^n} M(y,t)\, dy = 0,$$

then, for each $j \geq 1$,

$$\int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{R}^{n+1}_+ \setminus \widehat{B_{j+1}}}(y,t) M(y,t)\, dy = -\int_{\mathbb{R}^n} \mathbf{1}_{\widehat{B_{j+1}}}(y,t) M(y,t)\, dy, \quad \text{for a.e. } t > 0.$$

This, together with the fact that

$$\int_{\mathbb{R}^n} \mathbf{1}_{\widehat{B_{j+1}}}(y,t) M(y,t)\, dy = \mathbf{1}_{(0,r_{j+1})}(t) \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{B_{j+1}}}(y,t) M(y,t)\, dy,$$

gives, for a.e. $t > 0$,

$$\int_{\mathbb{R}^n} \mathbf{1}_{\widehat{B_{j+1}}}(y,t) M(y,t)\, dy = -\mathbf{1}_{(0,r_{j+1})}(t) \int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{R}^{n+1}_+ \setminus \widehat{B_{j+1}}}(y,t) M(y,t)\, dy.$$

Hence, for all $j \geq 1$,

$$\beta_j(x,t) = \mathbf{1}_{(0,r_{j+1})}(t) \int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{R}^{n+1}_+ \setminus \widehat{B_{j+1}}}(y,t) M(y,t)\, dy \left(\frac{\mathbf{1}_{B_{j+2}}(x)}{|B_{j+2}|} - \frac{\mathbf{1}_{B_{j+1}}(x)}{|B_{j+1}|} \right), \text{ for a.e. } t > 0.$$

Therefore,

$$\begin{aligned}
\left(\iint_{\widehat{B_{j+3}}} |\beta_j(x,t)|^r \frac{dx\,dt}{t} \right)^{\frac{1}{r}} &\lesssim \sum_{i \geq j+1} \frac{|B_{i+1}|}{|B_{j+1}|} \left(\iint_{\widehat{B_{j+3}}} \mathbf{1}_{B_{j+2}}(x) \left| \frac{1}{|B_{i+1}|} \int_{\mathbb{R}^n} \mathbf{1}_{\widehat{C_i}}(y,t) M(y,t)\, dy \right|^r \frac{dx\,dt}{t} \right)^{\frac{1}{r}} \\
&\lesssim \sum_{i \geq j+1} \left(\frac{|B_{i+1}|}{|B_{j+1}|} \right)^{1-\frac{1}{r}} \left(\iint_{\widehat{C_i}} |M(y,t)|^r \frac{dy\,dt}{t} \right)^{\frac{1}{r}} \\
&\lesssim \sum_{i \geq j+1} 2^{-(i+1)\varepsilon} |B_{j+1}|^{\frac{1}{r}-1} |B_{i+1}|^{1-\frac{1}{p}} \\
&\lesssim |B_{j+3}|^{\frac{1}{r}-\frac{1}{p}} \sum_{i \geq j+1} 2^{-(i+1)\varepsilon} \leq c' 2^{-j\varepsilon} |B_{j+3}|^{\frac{1}{r}-\frac{1}{p}},
\end{aligned}$$

where c' depends on ε, r, p only. Hence, $A'_j(x,t) := \frac{2^{j\varepsilon}}{c'}\beta_j$ is a $\mathfrak{T}^p_r$ atom.

Therefore, we have shown that $M = \sum_{j\geq 1} c2^{-j\varepsilon}A_j + \sum_{j\geq 1} c'2^{-j\varepsilon}A'_j$, which evidently shows that $M \in \mathfrak{T}^p_r$ with norm bounded by $(c+c')\left(\sum_{j\geq 1} |2^{-j\varepsilon p}|^p\right)^{\frac{1}{p}}$. □

Let us finally show that if $\mathcal{T}$ is a Calderón-Zygmund operator, then $\mathcal{T}$ applies $\mathfrak{T}_r^p$ atoms to $\mathfrak{T}_r^p$ molecules, up to a uniform constant. Note that, from the above proposition, and an adaptation of Lemma 2.32, this is enough to conclude the proof.

From part (c) of the proof, we already know that $\mathcal{T}$ applies $\mathfrak{T}_r^p$ atoms to T_r^p molecules, up to a uniform constant. It remains to show $\int_{\mathbb{R}^n} \mathcal{T}(A(\cdot,t))(x)\,dx = 0$. Note for almost every $t > 0$, $A(\cdot,t)$ is a multiple of an atom in the Hardy space $H^1(\mathbb{R}^n)$. Indeed, its support is contained in B, it is in $L^r(B)$ with $r > 1$ and has mean value 0. We know that $\mathcal{T}(A(\cdot,t)) \in L^1(\mathbb{R}^n)$ since $\mathcal{T}(A)$ has been shown to be a T_r^p molecule. Thus, $\int_{\mathbb{R}^n} \mathcal{T}(A(\cdot,t))(x)\,dx = 0$ as $\mathcal{T}^*(1) = 0$.

2.2.3 Riesz potentials and fractional maximal functions

For $0 < \alpha < n$, consider the Riesz potential

$$\mathcal{I}_\alpha(f)(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{1}{|x-z|^{n-\alpha}} f(z)\,dz,$$

where $\gamma(\alpha) = \pi^{\frac{n}{2}} 2^\alpha \Gamma(\alpha/2)/\Gamma\left(\frac{n-\alpha}{2}\right)$, and the fractional maximal function

$$\mathcal{M}_\alpha(f)(x) = \sup_{\tau>0} \tau^\alpha \fint_{B(x,\tau)} |f(y)|dy.$$

This operators act on tent spaces in the following way:

Theorem 2.34. *For $0 < \alpha < n$, $\frac{n}{n-\alpha} < r < \infty$, and $1 < p < q < \infty$ such that $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$,*

$$\mathcal{I}_\alpha, \mathcal{M}_\alpha : T_r^p \to T_r^q.$$

Before starting with the proof of the theorem, note that

$$\mathcal{M}_\alpha(f)(x) \le V_n^{-1} \mathcal{I}_\alpha(|f|)(x), \text{ for all } x \in \mathbb{R}^n, \tag{2.35}$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$. Consequently, it is enough to prove Theorem 2.34 for Riesz potentials. Let us start by proving the following pointwise inequality.

Lemma 2.36. *Let $0 < \alpha < n$, $1 < \vartheta < r < \infty$, and $\frac{\alpha}{n} = \frac{1}{\vartheta} - \frac{1}{r}$. Then, for any $x \in \mathbb{R}^n$, $t > 0$, if f is locally ϑ integrable,*

$$\left(\fint_{B(x,t)} |\mathcal{I}_\alpha(f)(y)|^r dy\right)^{\frac{1}{r}} \lesssim t^{n\left(\frac{1}{\vartheta}-\frac{1}{r}\right)} \left(\fint_{B(x,5t)} |f(y)|^\vartheta dy\right)^{\frac{1}{\vartheta}} + \mathcal{I}_\alpha\left(\fint_{B(\cdot,t)} |f(z)|dz\right)(x).$$

Proof. For each $x \in \mathbb{R}^n$ and $t > 0$, split the support of f into $B(x,5t)$ and $\mathbb{R}^n \setminus B(x,5t)$. Then,

$$\left(\fint_{B(x,t)} |\mathcal{I}_\alpha(f)(y)|^r dy\right)^{\frac{1}{r}} \le \left(\fint_{B(x,t)} |(\mathcal{I}_\alpha(\mathbf{1}_{B(x,5t)} f))(y)|^r dy\right)^{\frac{1}{r}} + \left(\fint_{B(x,t)} \left|\int_{|x-z|>5t} \frac{1}{|y-z|^{n-\alpha}} f(z)\,dz\right|^r dy\right)^{\frac{1}{r}} =: I + II.$$

On the one hand, using that $\mathcal{I}_\alpha : L^\vartheta(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ (see [75, Theorem 1, p.119]), we obtain that

$$I \lesssim t^{n\left(\frac{1}{\vartheta}-\frac{1}{r}\right)} \left(\fint_{B(x,5t)} |f(y)|^\vartheta dy\right)^{\frac{1}{\vartheta}}.$$

On the other hand,

$$
\begin{aligned}
II &\lesssim \left(\fint_{B(x,t)} \left(\int_{|x-z|>5t} \frac{1}{|y-z|^{n-\alpha}} |f(z)| \fint_{B(z,t)} d\xi \, dz \right)^r dy \right)^{\frac{1}{r}} \\
&\lesssim \left(\fint_{B(x,t)} \left(\int_{|x-\xi|>4t} \fint_{B(\xi,t)} \frac{1}{|y-z|^{n-\alpha}} |f(z)| \, dz \, d\xi \right)^r dy \right)^{\frac{1}{r}} \\
&\lesssim \left(\fint_{B(x,t)} \left(\int_{|x-\xi|>4t} \frac{1}{|x-\xi|^{n-\alpha}} \fint_{B(\xi,t)} \left(\frac{|x-\xi|}{|y-z|} \right)^{n-\alpha} |f(z)| \, dz \, d\xi \right)^r dy \right)^{\frac{1}{r}} \\
&\lesssim \left(\fint_{B(x,t)} \left(\int_{|x-\xi|>4t} \frac{1}{|x-\xi|^{n-\alpha}} \fint_{B(\xi,t)} |f(z)| \, dz \, d\xi \right)^r dy \right)^{\frac{1}{r}} \\
&= \int_{|x-\xi|>4t} \frac{1}{|x-\xi|^{n-\alpha}} \fint_{B(\xi,t)} |f(z)| \, dz \, d\xi \leq \mathcal{I}_\alpha \left(\fint_{B(\cdot,t)} |f(z)| dz \right)(x).
\end{aligned}
$$

□

Proof of Theorem 2.34.

Let $F \in T^p_r$. Taking $\vartheta = \frac{nr}{\alpha r + n}$ in Lemma 2.36, we obtain that

$$
\begin{aligned}
\|\mathcal{I}_\alpha(F)\|_{T^q_r} \lesssim & \left(\int_{\mathbb{R}^n} \left(\int_0^\infty t^{n(\frac{r}{\vartheta}-1)} \left(\fint_{B(x,5t)} |F(y,t)|^\vartheta dy \right)^{\frac{r}{\vartheta}} \frac{dt}{t} \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}} \\
& + \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \left(\mathcal{I}_\alpha \left(\fint_{B(\cdot,t)} |F(y,t)| dy \right)(x) \right)^r \frac{dt}{t} \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}} =: I + II.
\end{aligned}
$$

Since $r > \vartheta$, applying successively Jensen's inequality, [2, Theorem 2.19] for $s_1 = \frac{1}{r} - \frac{1}{\vartheta}$, $s_0 = 0$, $p_0 = p$, $p_1 = q$, and $q = r$, and [32, Section 3, Proposition 4] (we use this proposition for r instead of 2, but the proof is the same), we have

$$
\begin{aligned}
I &\lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,5t)} \left| t^{n(\frac{1}{\vartheta}-\frac{1}{r})} F(y,t) \right|^r \frac{dy\,dt}{t^{n+1}} \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}} \\
&\lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,5t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p}{r}} dx \right)^{\frac{1}{p}} \lesssim \|F\|_{T^p_r}.
\end{aligned}
$$

Finally, to estimate II, we shall proceed by extrapolation. We first recall some definitions. We say that a weight w is a $A_{\tau,s}$ weight, for $1 < \tau \leq s < \infty$, if it satisfies for every $B \subset \mathbb{R}^n$ that

$$
\left(\fint_B w(x)^s \, dx \right)^{\frac{1}{s}} \left(\fint_B w(x)^{-\tau'} \, dx \right)^{\frac{1}{\tau'}} \leq C.
$$

Now, since $0 < \alpha < n$ and $1 < \vartheta < \frac{n}{\alpha}$ with $\frac{1}{\vartheta} - \frac{1}{r} = \frac{\alpha}{n}$, by [71, Theorem 4] for all $w \in A_{\vartheta,r}$ we have that $\mathcal{I}_\alpha : L^\vartheta(w^\vartheta) \to L^r(w^r)$. This and Minkowski's integral inequality imply

$$
\left(\int_{\mathbb{R}^n} \int_0^\infty \left| \mathcal{I}_\alpha \left(\fint_{B(\cdot,t)} |F(y,t)| dy \right)(x) \right|^r \frac{dt}{t} w(x)^r dx \right)^{\frac{1}{r}}
$$

$$\lesssim \left(\int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{I}_\alpha \left(\fint_{B(\cdot,t)} |F(y,t)| dy \right)(x) \right|^r w(x)^r dx \frac{dt}{t} \right)^{\frac{1}{r}}$$

$$\lesssim \left(\int_0^\infty \left(\int_{\mathbb{R}^n} \left(\fint_{B(x,t)} |F(y,t)| dy \right)^\vartheta w(x)^\vartheta dx \right)^{\frac{r}{\vartheta}} \frac{dt}{t} \right)^{\frac{1}{r}}$$

$$\lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \left(\fint_{B(x,t)} |F(y,t)| dy \right)^r \frac{dt}{t} \right)^{\frac{\vartheta}{r}} w(x)^\vartheta dx \right)^{\frac{1}{\vartheta}}$$

$$\lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |F(y,t)|^r \frac{dy\, dt}{t^{n+1}} \right)^{\frac{\vartheta}{r}} w(x)^\vartheta dx \right)^{\frac{1}{\vartheta}}.$$

Then, since $1 < \vartheta < r < \infty$ and $1 < p < q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{1}{\vartheta} - \frac{1}{r}$, applying Theorem 1.46, part (*d*), we have that, for all $w_0 \in A_{p,q}$, and $F \in T^p_r$,

$$\left(\int_{\mathbb{R}^n} \left(\int_0^\infty \left(\mathcal{I}_\alpha \left(\fint_{B(\cdot,t)} |F(y,t)| dy \right)(x) \right)^r \frac{dt}{t} \right)^{\frac{q}{r}} w_0(x)^q dx \right)^{\frac{1}{q}}$$

$$\lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |F(y,t)|^r \frac{dy\, dt}{t^{n+1}} \right)^{\frac{p}{r}} w_0(x)^p dx \right)^{\frac{1}{p}}.$$

In particular for $w_0 \equiv 1$, we have that $w_0 \in A_{p,q}$. Hence,

$$II \lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |F(y,t)|^r \frac{dy\, dt}{t^{n+1}} \right)^{\frac{p}{r}} dx \right)^{\frac{1}{p}} = \|F\|_{T^p_r}.$$

□

2.2.4 Riesz transform

We have the following result regarding the operator L defined in Section 1.2.

Theorem 2.37. *Let $L = -\operatorname{div}(A\nabla)$ be an elliptic operator with complex-valued coefficients. For $q_-(L) < p, r < q_+(L)$ we have*

$$\nabla L^{-\frac{1}{2}} : T^p_r \to T^p_r.$$

In oder to prove this theorem, we shall obtain a pointwise inequality for the Riesz transform taking a generalized version of two inequalities that appear in [3, Lemma 4.8 and (4.6)]. These are:

Lemma 2.38. *For every ball B, with radius r_B, and $q_-(L) < r < q_+(L)$,*

$$\|\nabla L^{-\frac{1}{2}}(I - e^{-r_B^2 L})^M h\|_{L^r(B)} \le |B|^{\frac{1}{r}} \sum_{j \ge 1} g(j) \left(\fint_{2^{j+1}B} |h|^r \right)^{\frac{1}{r}},$$

with $g(j) = C 2^{j\frac{n}{2}} 4^{-jM}$, where $M \in \mathbb{N}$ is arbitrary and C depends on M.

Lemma 2.39. *For every ball B, with radius r_B, any constant $k > 0$, and $q_-(L) < p_0 \le r < q_+(L)$,*

$$\left(\fint_B |\nabla e^{-k r_B^2 L} h|^r \right)^{\frac{1}{r}} \le \sum_{j \ge 1} g(j) \left(\fint_{2^{j+1}B} |\nabla h|^{p_0} \right)^{\frac{1}{p_0}},$$

with $\sum_{j \ge 1} g(j) < \infty$.

It is in the first inequality that the integral representation $\nabla L^{-\frac{1}{2}}h = \pi^{-1/2}\int_0^\infty \nabla e^{-tL}h\frac{dt}{\sqrt{t}}$ was used for appropriate h (in order to replace the kernel representation in the case of Calderón-Zygmund operators).

From these two results we have the following corollary.

Corollary 2.40. *Let $q_-(L) < p_0 < r < q_+(L)$. For every $x \in \mathbb{R}^n$ and $t > 0$ and $f \in L^r(\mathbb{R}^n)$.*

$$\left(\fint_{B(x,t)} |\nabla L^{-\frac{1}{2}}(f)(y)|^r\, dy\right)^{\frac{1}{r}} \lesssim \sum_{j\geq 1} 4^{-jM}\left(\int_{B(x,2^{j+1}t)} |f(y)|^r \frac{dy}{t^n}\right)^{\frac{1}{r}} + \sum_{k=1}^{M} C_{k,M}\mathcal{M}_{p_0}\left(\nabla L^{-\frac{1}{2}}e^{-\frac{kt^2}{2}L}(f)\right)(x),$$

where $M \in \mathbb{N}$ is arbitrarily large and $\mathcal{M}_{p_0}(f) := (\mathcal{M}(|f|^{p_0}))^{\frac{1}{p_0}}$.

Proof. Fix $x \in \mathbb{R}^n$, $t > 0$ and $M \in \mathbb{N}$ arbitrarily large. We have that

$$\begin{aligned}
&\left(\fint_{B(x,t)} |\nabla L^{-\frac{1}{2}}(f)(y)|^r\, dy\right)^{\frac{1}{r}} \\
&\qquad \lesssim \left(\fint_{B(x,t)} |\nabla L^{-\frac{1}{2}}(I - e^{-t^2L})^M(f)(y)|^r\, dy\right)^{\frac{1}{r}} + \left(\fint_{B(x,t)} |\nabla L^{-\frac{1}{2}}A_{t,M}(f)(y)|^r\, dy\right)^{\frac{1}{r}} =: I + II,
\end{aligned}$$

where $A_{t,M} := I - (I - e^{-t^2L})^M$. Then, applying Lemma 2.38 for $B = B(x,t)$ and $h = f$, we obtain that

$$I \lesssim \sum_{j\geq 1} 4^{-jM}\left(\int_{B(x,2^{j+1}t)} |f(y)|^r \frac{dy}{t^n}\right)^{\frac{1}{r}}.$$

As for the estimate of II, note that expanding the binomial expression, we have that $A_{t,M} = \sum_{k=1}^M C_{k,M}e^{-kt^2L}$. Then, applying Lemma 2.39 for $B = B(x,t)$ and $h = L^{-\frac{1}{2}}e^{-\frac{kt^2}{2}L}f$,

$$\begin{aligned}
II &\lesssim \sum_{k=1}^{M} C_{k,M}\left(\fint_{B(x,t)} |\nabla e^{-\frac{kt^2L}{2}}L^{-\frac{1}{2}}e^{-\frac{kt^2L}{2}}(f)(y)|^r\, dy\right)^{\frac{1}{r}} \\
&\lesssim \sum_{k=1}^{M} C_{k,M}\sum_{j\geq 1} g(j)\left(\fint_{B(x,2^{j+1}t)} |\nabla L^{-\frac{1}{2}}e^{-\frac{kt^2}{2}L}(f)(y)|^{p_0}dy\right)^{\frac{1}{p_0}} \\
&\lesssim \sum_{k=1}^{M} C_{k,M}\mathcal{M}_{p_0}\left(\nabla L^{-\frac{1}{2}}e^{-\frac{kt^2}{2}L}(f)\right)(x).
\end{aligned}$$

□

Proof of Theorem 2.37.

Recall that the Riesz transform associated with this operator L, acting over a function $F \in T^r_r$ (so that $F(\cdot,t) \in L^r(\mathbb{R}^n)$ for almost every $t > 0$), is defined by $\nabla L^{-\frac{1}{2}}(F(\cdot,t))(x)$ for almost every $t > 0$. Applying Corollary 2.40, we obtain, for all $F \in T^r_r$,

$$\begin{aligned}
\|\nabla L^{-\frac{1}{2}}(F)\|_{T^p_r} &\lesssim \sum_{j\geq 1} 4^{-jM}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,2^{j+1}t)} |F(y,t)|^r\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} \\
&\quad + \sum_{k=1}^{M} C_{k,M}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty \left|\mathcal{M}_{p_0}\left(\nabla L^{-\frac{1}{2}}e^{-\frac{kt^2}{2}L}(F(\cdot,t))\right)(x)\right|^r\frac{dt}{t}\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} =: I + \sum_{k=0}^{M} C_{k,M}II.
\end{aligned}$$

Applying Theorem 2.5, and taking $M > \frac{n}{\min\{p,r\}}$, we have that

$$I \lesssim \sum_{j\geq 1} 4^{-j\left(M-\frac{n}{\min\{p,r\}}\right)} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} \lesssim \|F\|_{T^p_r}.$$

Finally the estimate of II follows by extrapolation. For all weights $w \in A_{\frac{r}{q_-(L)}} \cap RH_{\left(\frac{q_+(L)}{r}\right)'}$ we have that $\nabla L^{-\frac{1}{2}} : L^r(w) \to L^r(w)$ ([11, Theorem 5.2]) and that $\mathcal{M}_{p_0} : L^r(w) \to L^r(w)$, for some $p_0 > q_-(L)$ close enough to $q_-(L)$ so that $w \in A_{\frac{r}{p_0}}$. Besides, we can also take $r < q_0 < q_+(L)$ so that $w \in RH_{\left(\frac{q_0}{r}\right)'}$. Using these three facts, applying Hölder's inequality for $\frac{q_0}{r}$, the $L^r(\mathbb{R}^n) - L^{q_0}(\mathbb{R}^n)$ off-diagonal estimates that the semigroup $\{e^{-t^2L}\}_{t>0}$ satisfies (see [3]), and Fubini's theorem, we have that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \int_0^\infty |\mathcal{M}_{p_0}(\nabla L^{-\frac{1}{2}} e^{-\frac{kt^2}{2}L}(F(\cdot,t)))(x)|^r \frac{dt}{t} w(x)dx\right)^{\frac{1}{r}} \\
&\quad\lesssim \left(\int_0^\infty \int_{\mathbb{R}^n} |e^{-\frac{kt^2}{2}L}(F(\cdot,t))(x)|^r w(x)dx \frac{dt}{t}\right)^{\frac{1}{r}} \\
&\quad= \left(\int_0^\infty \int_{\mathbb{R}^n} |e^{-\frac{kt^2}{2}L}(F(\cdot,t))(x)|^r \int_{B(x,t)} w(y)\frac{dy}{w(B(x,t))} w(x)dx \frac{dt}{t}\right)^{\frac{1}{r}} \\
&\quad= \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(y,t)} |e^{-\frac{kt^2}{2}L}(F(\cdot,t))(x)|^r w(x)\frac{dx}{w(B(x,t))} \frac{dt}{t} w(y)dy\right)^{\frac{1}{r}} \\
&\quad\lesssim \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(y,t)} |e^{-\frac{kt^2}{2}L}(F(\cdot,t))(x)|^r w(x)dx \frac{dt}{tw(B(y,t))} w(y)dy\right)^{\frac{1}{r}} \\
&\quad\lesssim \left(\int_{\mathbb{R}^n} \int_0^\infty \left(\int_{B(y,t)} |e^{-\frac{kt^2}{2}L}(F(\cdot,t)(x)|^{q_0}dx\right)^{\frac{r}{q_0}} \left(\int_{B(y,t)} w(x)^{\left(\frac{q_0}{r}\right)'}dx\right)^{\frac{q_0-r}{q_0}} \frac{dt}{tw(B(y,t))} w(y)dy\right)^{\frac{1}{r}} \\
&\quad\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(y,2^{j+1}t)} |F(x,t)|^r dx \left(\fint_{B(y,t)} w(x)^{\left(\frac{q_0}{r}\right)'}dx\right)^{\frac{q_0-r}{q_0}} \frac{dt}{tw(B(y,t))} w(y)dy\right)^{\frac{1}{r}} \\
&\quad\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(y,2^{j+1}t)} |F(x,t)|^r \frac{dx\,dt}{t^{n+1}} w(y)dy\right)^{\frac{1}{r}} \\
&\quad\lesssim \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(y,t)} |F(x,t)|^r \frac{dx\,dt}{t^{n+1}} w(y)dy\right)^{\frac{1}{r}}.
\end{aligned}$$

The second inequality follows from the fact that $B(y,t) \subset B(x,2t)$ if $x \in B(y,t)$ and from the doubling property of the weight. Then, $w(B(y,t)) \leq w(B(x,2t) \leq 2^{nc_w} w(B(x,t))$.

Therefore, we have that, for all $w \in A_{\frac{r}{q_-(L)}} \cap RH_{\left(\frac{q_+(L)}{r}\right)'}$ and $F \in T^r_r$,

$$\int_{\mathbb{R}^n} \left(\int_0^\infty |\mathcal{M}_{p_0}(\nabla L^{-\frac{1}{2}} e^{-\frac{kt^2}{2}L}(F(\cdot,t)))(x)|^r \frac{dt}{t}\right)^{\frac{r}{r}} w(x)dx \lesssim \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{r}{r}} w(x)dx.$$

Recall that p_0 depends on w. But if we now fix $p_0 > q_-(L)$, we have this inequality for all $w \in A_{\frac{r}{p_0}} \cap RH_{\left(\frac{q_+(L)}{r}\right)'}$. Then, applying Theorem 1.46, part (c), we obtain that, for all $p_0 < p < q_+(L)$, $w_0 \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_+(L)}{p}\right)'}$, and all $F \in T^r_r$,

$$\int_{\mathbb{R}^n} \left(\int_0^\infty |\mathcal{M}_{p_0}(\nabla L^{-\frac{1}{2}} e^{-\frac{kt^2}{2}L}(F(\cdot,t)))(x)|^r \frac{dt}{t}\right)^{\frac{p}{r}} w_0(x)dx \lesssim \int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{r}} w_0(x)dx.$$

In particular, if we take $w_0 \equiv 1$, we have that $w_0 \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_+(L)}{p}\right)'}$. Then, for all $p_0 < r, p < q_+(L)$ and $1 \le k \le M$, we finally conclude that

$$II \lesssim \|F\|_{T^p_r}.$$

In conclusion, we obtain $\|\nabla L^{-\frac{1}{2}}(F)\|_{T^p_r} \lesssim \|F\|_{T^p_r}$ for all $p_0 < r, p < q_+(L)$ and all $q_-(L) < p_0 < q_+(L)$, and for all $F \in T^r_r$. The density of $T^r_r \cap T^p_r$ in T^p_r finishes the proof. □

2.2.5 Some remarks

We note that all the arguments using extrapolation prove much more than what we stated.

For $\frac{n}{n+1} < q < \infty$ and $1 < r < \infty$, one can show that the set

$$E := \left\{\varphi \in C^\infty_0(\mathbb{R}^{n+1}_+) : \int_{\mathbb{R}^n} \varphi(x,t)\, dx = 0 \text{ for all } t > 0\right\}$$

is dense in $\mathfrak{T}^q_r$ when $q \le 1$ and in T^q_r when $q > 1$. For $q \le 1$, it suffices to do that on $\mathfrak{T}^q_r$ atoms and for $q > 1$, we already know that the space of compactly supported smooth functions in $\mathbb{R}^{n+1}_+$ is dense and those functions can be approximated in L^r norm imposing the mean value condition using $r > 1$. So the fact that there is a common dense subspace is an indication that the space $\mathfrak{T}^q_r$ is not too small.

It is clear that one can push Theorem 2.25, part (c) and (d), to any Calderón-Zygmund operator on $\mathbb{R}^n$ of order $\delta \ge 1$ (see [37], [50] for definition) imposing more vanishing moments in the definition of $\mathfrak{T}^q_r$ atoms when $q \le \frac{n}{n+1}$ and more cancellation conditions on the adjoint. Similarly, we can play the same game on slice-spaces. These slice-spaces will be subspaces of the classical real Hardy spaces as one can show. We do not insist.

Consider a standard Littlewood-Paley decomposition of $\mathbb{R}^n$ given from a pair of C^∞_0 functions $\psi, \tilde{\psi}$ with all vanishing moments and such that

$$\int_0^\infty Q_t \tilde{Q}_t f \frac{dt}{t} = f$$

on appropriate distributions f, where Q_t and $\tilde{Q}_t$ are convolutions with ψ_t and $\tilde{\psi}_t$ respectively. We have set $\psi_t(x) = t^{-n}\psi(x/t)$ and likewise for $\tilde{\psi}_t$. One can show that $f \in H^q(\mathbb{R}^n)$ implies $F(x,t) = \tilde{Q}_t f(x)$ belongs to $\mathfrak{T}^q_2$ and the action is bounded. Conversely, $F \in \mathfrak{T}^q_2$ implies that $f = \int_0^\infty Q_t F(\cdot,t)\, \frac{dt}{t}$ belongs to $H^q(\mathbb{R}^n)$ and the action is bounded. This is fairly easy to show using atoms and molecules. This can be done for $0 < q \le 1$. Thus, $H^q(\mathbb{R}^n)$ can be seen as a retract of the space $\mathfrak{T}^q_2$. It is also the case using T^q_2 instead as shown in [32]. Nevertheless, the spaces $\mathfrak{T}^q_2$ are preserved by the singular integrals (of convolution) while the T^q_2 are not. It would be interesting to explore further these spaces (interpolation, etc) and their applications. In particular, one could recover boundedness for Calderón-Zygmund operators on tent spaces from interpolation.

2.3 Weighted tent spaces

Recall that the tent space T^p_r was defined as the set of all the measurable functions F such that the operator $\mathcal{A}_r$ applied to those functions is on $L^p(\mathbb{R}^n)$. Now, consider w an A_∞ weight, then, for all $0 < p, r < \infty$, we define the weighted tent space $T^p_r(w)$ as the set of all measurable functions such that $\mathcal{A}_r F \in L^p(w)$ (these spaces are also defined in [27]). Analogously as in the unweighted case, we can obtain that, for all $1 \le p < \infty$ and $1 \le r < \infty$, these spaces endowed with the norm $\|F\|_{T^p_r(w)} := \|\mathcal{A}_r F\|_{L^p(w)}$, are quasi-Banach spaces, (we do not consider the case $r = \infty$ because we shall not use it). To see this we need an analogous inequality to (2.2), and then, argue as in the unweighted case. This is, for every K compact subset in $\mathbb{R}^{n+1}_+$, we have that, for some constants $c_1, c_2, c_K > 0$, and $x_K \in \mathbb{R}^n$,

$$K \subset \{(y,t) \in \mathbb{R}^{n+1}_+ : c_1 < t < c_2,\ y \in B(x_K, c_K)\}.$$

Then,

$$\left(\iint_K |F(y,t)|^r dy\,dt\right)^{\frac{1}{r}} \le [w]_{A_{\widehat{r}}} c_2 c_1^{-(n+1)/r'} \left(\frac{c_K + c_2}{c_1}\right)^{n\widehat{r}} w(B(x_K, c_K))^{-1/p} \|\mathcal{A}_r F\|_{L^p(w)}.$$

In order to obtain this inequality, we use that $w \in A_\infty$ implies for some $\widehat{r} \ge 1$ that $w \in A_{\widehat{r}}$ and hence, by (1.10), we have

$$\begin{aligned}
&\left(\iint_K |F(y,t)|^r dy\,dt\right)^{\frac{1}{r}} \le \sup_{\|\phi\|_{L^{r'}(K)}\le 1} \iint_K |F(y,t)||\phi(y,t)|dy\,dt \\
&\le c_2^{n+1} \sup_{\|\phi\|_{L^{r'}(K)}\le 1} \iint_K |F(y,t)||\phi(y,t)| \fint_{B(y,t)} dw \frac{dy\,dt}{t^{n+1}} \\
&\le [w]_{A_{\widehat{r}}} c_2^{n+1} \left(\frac{c_K + c_2}{c_1}\right)^{n\widehat{r}} w(B(x_K, c_K + c_2))^{-1} \sup_{\|\phi\|_{L^{r'}(K)}\le 1} \int_{B(x_K, c_K + c_2)} \iint_K \mathbf{1}_{B(x,t)}(y)|F(y,t)||\phi(y,t)| \frac{dy\,dt}{t^{n+1}} w(x)dx \\
&\le [w]_{A_{\widehat{r}}} c_2^{n+1} c_1^{-(n+1)/r'} \left(\frac{c_K + c_2}{c_1}\right)^{n\widehat{r}} w(B(x_K, c_K + c_2))^{-1} \\
&\qquad \times \sup_{\|\phi\|_{L^{r'}(K)}\le 1} \int_{B(x_K, c_K + c_2)} \left(\int_{c_1}^{c_2} \int_{B(x,t)\cap B(x_K, c_K)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{r}} w(x)dx\,\|\phi\|_{L^{r'}(K)} \\
&\le [w]_{A_{\widehat{r}}} c_2 c_1^{-(n+1)/r'} \left(\frac{c_K + c_2}{c_1}\right)^{n\widehat{r}} w(B(x_K, c_K + c_2))^{-1/p} \|\mathcal{A}_r F\|_{L^p(w)}. \qquad (2.41)
\end{aligned}$$

Remark 2.42. *By the previous inequality, reasoning as in the unweighted case, we have that the functions in $L^r(\mathbb{R}^{n+1}_+)$ with compact support are dense in $T^p_r(w)$, for all $0 < p, q < \infty$.*

2.3.1 Change of angles

As in the unweighted case, we also have that the definition of the weighted tent spaces $T^p_r(w)$ does not depend on the aperture of the cone Γ^α used to define the operator $\mathcal{A}^\alpha_r$. We prove this for $r = 2$ in the next proposition. For a general r, the result is stated in Proposition 2.74, and it follows from the equality $\mathcal{A}^\alpha_r F(x) = \mathcal{A}^\alpha(|F|^{\frac{r}{2}})(x)^{\frac{2}{r}}$, and the case $r = 2$.

Proposition 2.43. *Let $0 < \alpha \le \beta < \infty$.*

(*i*) *For every $w \in A_{\widehat{r}}$, $1 \le \widehat{r} < \infty$, there holds*

$$\|\mathcal{A}^\beta F\|_{L^p(w)} \le C \left(\frac{\beta}{\alpha}\right)^{\frac{n\widehat{r}}{p}} \|\mathcal{A}^\alpha F\|_{L^p(w)} \quad \textit{for all} \quad 0 < p \le 2\widehat{r}.$$

(*ii*) *For every $w \in RH_{s'}$, $1 \le s < \infty$, there holds*

$$\|\mathcal{A}^\alpha F\|_{L^p(w)} \le C \left(\frac{\alpha}{\beta}\right)^{\frac{n}{sp}} \|\mathcal{A}^\beta F\|_{L^p(w)} \quad \textit{for all} \quad \frac{2}{s} \le p < \infty.$$

In Remark 2.56 below we shall show that for power weights $w_\theta(x) := |x|^\theta$, the previous estimates are sharp: the exponents $n\widehat{r}/p$ in (*i*) and n/sp in (*ii*) cannot be improved. This should be compared with [4] where the unweighted case was considered (see also [66] and [27]).

Proof of (*i*). We first observe that if $0 < \alpha \le \beta < \infty$ then $\mathcal{A}^\beta F(x) = \mathcal{A}^{\beta/\alpha} \widetilde{F}$, where $\widetilde{F}(x,t) = \alpha^{\frac{n}{2}} F(x, t/\alpha)$. Thus, we can reduce matters to obtaining that for every $\alpha \ge 1$ and for every $w \in A_{\widehat{r}}$, $1 \le \widehat{r} < \infty$, there holds

$$\|\mathcal{A}^\alpha F\|_{L^p(w)} \le C\alpha^{\frac{n\widehat{r}}{p}} \|\mathcal{A} F\|_{L^p(w)}, \quad \text{for all} \quad 0 < p \le 2\widehat{r}. \qquad (2.44)$$

We then prove (2.44) by splitting the proof into three steps. We first obtain the case $p = 2$ and $1 \le \widehat{r} < \infty$. From this, we extrapolate concluding the desired estimate in the ranges $0 < p \le 2\widehat{r}$ and $1 < \widehat{r} < \infty$. Finally, we will consider the case $\widehat{r} = 1$ and $0 < p < 2$.

Fix from now on $\alpha > 1$. For the first step, let $p = 2$ and $w \in A_{r_0}$, $1 \le r_0 < \infty$. From (1.10), we easily obtain

$$\begin{aligned}\|\mathcal{A}^\alpha F\|_{L^2(w)} &= \left(\int_{\mathbb{R}^n}\int_0^\infty \int_{|x-y|<\alpha t} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx\right)^{\frac12} = \left(\int_{\mathbb{R}^n}\int_0^\infty |F(y,t)|^2 w(B(y,\alpha t))\frac{dy\,dt}{t^{n+1}}\right)^{\frac12}\\ &\lesssim \alpha^{\frac{nr_0}{2}}\left(\int_{\mathbb{R}^n}\int_0^\infty |F(y,t)|^2 w(B(y,t))\frac{dy\,dt}{t^{n+1}}\right)^{\frac12} = \alpha^{\frac{nr_0}{2}}\left(\int_{\mathbb{R}^n}\int_0^\infty \int_{|x-y|<t} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx\right)^{\frac12}\\ &= \alpha^{\frac{nr_0}{2}}\|\mathcal{A}F\|_{L^2(w)}. \end{aligned} \tag{2.45}$$

We shall extrapolate from this inequality. To set the stage, take an arbitrary $1 \le r_0 < \infty$ and consider $\mathcal{F}$ the family of pairs $(f,g) = \big((\mathcal{A}^\alpha F)^{\frac{2}{r_0}}, \alpha^n(\mathcal{A}F)^{\frac{2}{r_0}}\big)$. Notice that (2.45) immediately gives that for every $w \in A_{r_0}$

$$\int_{\mathbb{R}^n} f(x)^{r_0}\, w(x)\,dx = \int_{\mathbb{R}^n} \mathcal{A}^\alpha F(x)^2\, w(x)\,dx \le C\,\alpha^{n\,r_0}\int_{\mathbb{R}^n} \mathcal{A}F(x)^2\, w(x)\,dx = C\int_{\mathbb{R}^n} g(x)^{r_0}\, w(x)\,dx,$$

where C does not depend on α. Next, we apply (a) in Theorem 1.46 to conclude that for every $1 < \widehat{r} < \infty$ and for every $w \in A_{\widehat{r}}$

$$\int_{\mathbb{R}^n} \mathcal{A}^\alpha F(x)^{\frac{2\widehat{r}}{r_0}}\, w(x)\,dx = \int_{\mathbb{R}^n} f(x)^{\widehat{r}}\, w(x)\,dx \le C\int_{\mathbb{R}^n} g(x)^{\widehat{r}}\, w(x)\,dx = C\,\alpha^{n\,r}\int_{\mathbb{R}^n} \mathcal{A}F(x)^{\frac{2\widehat{r}}{r_0}}\, w(x)\,dx,$$

where C does not depend on α. From this, using that $1 \le r_0 < \infty$ is arbitrary, we conclude (2.44) under the restriction $1 < \widehat{r} < \infty$.

To complete the proof it remains to consider the case $\widehat{r} = 1$, (i.e., $w \in A_1$) and $0 < p < 2$. Notice that if $\|\mathcal{A}F\|_{L^p(w)} = \infty$ the inequality follows immediately. So, we can assume that $\|\mathcal{A}F\|_{L^p(w)} < \infty$.

For a fixed $\lambda > 0$, set

$$E_\lambda := \{x \in \mathbb{R}^n : \mathcal{A}F(x) \le \lambda\}, \qquad O_\lambda := \mathbb{R}^n \backslash E_\lambda = \{x \in \mathbb{R}^n : \mathcal{A}F(x) > \lambda\}.$$

Then, for each $0 < \gamma < 1$, we also consider the set of global γ-density with respect to E_λ defined by

$$E_\lambda^* := \left\{x \in \mathbb{R}^n : \frac{|E_\lambda \cap B|}{|B|} \ge \gamma,\ \forall B \text{ centered at } x\right\}$$

and denote its complement by

$$O_\lambda^* = \left\{x \in \mathbb{R}^n : \exists\, r > 0 \text{ such that } \frac{|O_\lambda \cap B(x,r)|}{|B(x,r)|} > 1-\gamma\right\} = \left\{x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_\lambda})(x) > 1-\gamma\right\}, \tag{2.46}$$

where recall that $\mathcal{M}$ is the centered Hardy-Littlewood maximal operator.

Note that if $x_k \to x$ then $\mathbf{1}_{\Gamma(x_k)}(y,t) \to \mathbf{1}_{\Gamma(x)}(y,t)$ for a.e. $(y,t) \in \mathbb{R}^{n+1}_+$. This and the Fatou Lemma clearly imply that E_λ is closed. We next show that, for each $0 < \gamma < 1$, E_λ^* is a nonempty closed set contained in E_λ. Notice that the fact that $\mathcal{M} : L^1(w) \to L^{1,\infty}(w)$, since $w \in A_1$, and our earlier assumption ($\|\mathcal{A}F\|_{L^p(w)} < \infty$) give

$$w(O_\lambda^*) = w\left(\left\{x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_\lambda})(x) > 1-\gamma\right\}\right) \lesssim \frac{1}{1-\gamma} w(O_\lambda) \le \frac{1}{(1-\gamma)\,\lambda^p}\|\mathcal{A}F\|^p_{L^p(w)} < \infty.$$

This immediately implies that E_λ^* cannot be empty.

Next, we see that $E_\lambda^* \subset E_\lambda$, for all $0 < \gamma < 1$. This follows from the fact that E_λ is closed: if $x \notin E_\lambda$, there exists $r > 0$ such that $B(x,r) \cap E_\lambda = \emptyset$, and then $x \notin E_\lambda^*$.

Finally, we show that E^*_λ is closed. Let $\{x_k\}_k \subset E^*_\lambda$ be such that $x_k \to x$. Take an arbitrary $r > 0$ and define the functions $f_k = \mathbf{1}_{E_\lambda \cap B(x_k,r)}$ which satisfy $f_k \to \mathbf{1}_{E_\lambda \cap B(x,r)}$ a.e. in $\mathbb{R}^n$. Note also that for k large enough $f_k \le \mathbf{1}_{B(x,2r)}$ (since $x_k \in B(x,r)$). Thus, by the Dominated Convergence Theorem, we conclude that

$$|E_\lambda \cap B(x,r)| = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(y)\,dy = \lim_{k\to\infty} |E_\lambda \cap B(x_k,r)|.$$

On the other hand, since $x_k \in E^*_\lambda$ we have that $|E_\lambda \cap B(x_k,r)| \ge \gamma|B(x_k,r)| = \gamma|B(x,r)|$. This in turn implies that for every $r > 0$

$$\frac{|E_\lambda \cap B(x,r)|}{|B(x,r)|} \ge \gamma,$$

which yields that $x \in E^*_\lambda$ and hence E^*_λ is closed.

After these preparations, given $(y,t) \in \mathcal{R}^\alpha(E^*_\lambda)$, there exists $\bar{x} \in E^*_\lambda$ such that $|\bar{x} - y| < \alpha t$. Therefore, for $z = y - \frac{t}{2}\frac{y-\bar{x}}{|y-\bar{x}|}$ we have that $B\left(z, \frac{t}{2}\right) \subset B(\bar{x},\alpha t) \cap B(y,t)$ and

$$|B(\bar{x},\alpha t) \setminus B(y,t)| \le \left|B(\bar{x},\alpha t) \setminus B\left(z,\frac{t}{2}\right)\right| = |B(\bar{x},\alpha t)| - \left|B\left(z,\frac{t}{2}\right)\right| = |B(\bar{x},\alpha t)| \left(1 - \frac{1}{2^n\alpha^n}\right) = c_\alpha\, |B(\bar{x},\alpha t)|,$$

with $c_\alpha = \left(1 - \frac{1}{2^n\alpha^n}\right) < 1$. This and the fact that $\bar{x} \in E^*_\lambda$ yield

$$\begin{aligned}\gamma|B(\bar{x},\alpha t)| \le |E_\lambda \cap B(\bar{x},\alpha t)| &= |E_\lambda \cap B(\bar{x},\alpha t) \setminus B(y,t)| + |E_\lambda \cap B(\bar{x},\alpha t) \cap B(y,t)| \\ &\le c_\alpha\, |B(\bar{x},\alpha t)| + |E_\lambda \cap B(y,t)|.\end{aligned}$$

Choosing $\gamma = \frac{1+c_\alpha}{2}$ we conclude that

$$|E_\lambda \cap B(y,t)| \ge \frac{1}{2^{n+1}\alpha^n}|B(\bar{x},\alpha t)| = \frac{1}{2^{n+1}\alpha^n}|B(y,\alpha t)|. \tag{2.47}$$

From this and (1.10), we have for every $(y,t) \in \mathcal{R}^\alpha(E^*_\lambda)$,

$$\frac{w(E_\lambda \cap B(y,t))}{w(B(y,\alpha t))} \ge [w]_{A_1}^{-1} \frac{|E_\lambda \cap B(y,t)|}{|B(y,\alpha t)|} \ge \frac{1}{2^{n+1}\alpha^n [w]_{A_1}}. \tag{2.48}$$

We use this to show that

$$\begin{aligned}\int_{E^*_\lambda} \mathcal{A}^\alpha F(x)^2 w(x)\,dx &= \int_{E^*_\lambda} \int_0^\infty \int_{\mathbb{R}^n} |F(y,t)|^2 \mathbf{1}_{B(0,1)}\left(\frac{x-y}{\alpha t}\right) w(x) \frac{dy\,dt}{t^{n+1}}\,dx \\ &\le \iint_{\mathcal{R}^\alpha(E^*_\lambda)} |F(y,t)|^2 \int_{B(y,\alpha t)} w(x)\,dx \frac{dy\,dt}{t^{n+1}} \\ &\le \alpha^n [w]_{A_1} \iint_{\mathcal{R}^\alpha(E^*_\lambda)} |F(y,t)|^2 \int_{B(y,t)\cap E_\lambda} w(x)\,dx \frac{dy\,dt}{t^{n+1}} \\ &\le \alpha^n [w]_{A_1} \int_{E_\lambda} \mathcal{A}F(x)^2 w(x)\,dx.\end{aligned} \tag{2.49}$$

Therefore, from (2.49), (2.46), and the fact that $\mathcal{M} : L^1(w) \to L^{1,\infty}(w)$ (because $w \in A_1$), we obtain

$$\begin{aligned}w(\{x : \mathcal{A}^\alpha F(x) > \lambda\}) &\le w(\{x \in O^*_\lambda : \mathcal{A}^\alpha F(x) > \lambda\}) + w(\{x \in E^*_\lambda : \mathcal{A}^\alpha F(x) > \lambda\}) \\ &\le w(\{x : \mathcal{M}(\mathbf{1}_{O_\lambda})(x) > 1-\gamma\}) + \frac{1}{\lambda^2} \int_{E^*_\lambda} \mathcal{A}^\alpha F(x)^2 w(x)\,dx \\ &\lesssim \alpha^n [w]_{A_1} w(O_\lambda) + \alpha^n [w]_{A_1} \frac{1}{\lambda^2} \int_{E_\lambda} \mathcal{A}F(x)^2 w(x)\,dx\end{aligned}$$

$$= \alpha^n [w]_{A_1} w(\{x : \mathcal{A}F(x) > \lambda\}) + \alpha^n [w]_{A_1} \frac{1}{\lambda^2} \int_{E_\lambda} \mathcal{A}F(x)^2 w(x)\, dx.$$

Using this and that $0 < p < 2$ it follows that

$$\begin{aligned}
\|\mathcal{A}^\alpha F\|_{L^p(w)}^p &= \int_0^\infty p\, \lambda^p\, w(\{x : \mathcal{A}^\alpha F(x) > \lambda\}) \frac{d\lambda}{\lambda} \\
&\lesssim \alpha^n [w]_{A_1} \left(\int_0^\infty p\, \lambda^p\, w(\{x : \mathcal{A}F(x) > \lambda\}) \frac{d\lambda}{\lambda} + \int_0^\infty p \lambda^{p-2} \int_{E_\lambda} \mathcal{A}F(x)^2 w(x)\, dx \frac{d\lambda}{\lambda} \right) \\
&\le \alpha^n [w]_{A_1} \left(\|\mathcal{A}F\|_{L^p(w)}^p + \int_{\mathbb{R}^n} \mathcal{A}F(x)^2 \int_{\mathcal{A}F(x)}^\infty p \lambda^{p-2} \frac{d\lambda}{\lambda} w(x)\, dx \right) \\
&= C\, \alpha^n [w]_{A_1} \|\mathcal{A}F\|_{L^p(w)}^p.
\end{aligned}$$

This completes the proof of (*i*). □

Proof of (*ii*). As before, we can reduce matters to showing that for every $\alpha \ge 1$ and for every $w \in RH_{s'}$, $1 \le s < \infty$, there holds

$$\|\mathcal{A}F\|_{L^p(w)} \le C \alpha^{-\frac{n}{sp}} \|\mathcal{A}^\alpha F\|_{L^p(w)}, \quad \text{for all} \quad \frac{2}{s} \le p < \infty. \tag{2.50}$$

We show this estimate considering three cases: $p = 2$ and $1 \le s < \infty$, $2/s \le p < \infty$ and $1 < s < \infty$, and $s = 1$ and $2 < p < \infty$.

We start by taking $p = 2$ and $w \in RH_{s_0'}$ with $1 \le s_0 < \infty$. We proceed as in (2.45) and use (1.12) to obtain

$$\begin{aligned}
\|\mathcal{A}F\|_{L^2(w)} &= \left(\int_{\mathbb{R}^n} \int_0^\infty |F(y,t)|^2 w(B(y,t)) \frac{dy\, dt}{t^{n+1}} \right)^{\frac12} \\
&\lesssim \alpha^{-\frac{n}{2 s_0}} \left(\int_{\mathbb{R}^n} \int_0^\infty |F(y,t)|^2 w(B(y,\alpha t)) \frac{dy\, dt}{t^{n+1}} \right)^{\frac12} = \alpha^{-\frac{n}{2 s_0}} \|\mathcal{A}^\alpha F\|_{L^2(w)}.
\end{aligned} \tag{2.51}$$

For the second case we shall extrapolate from (2.51). Take an arbitrary $1 \le s_0 < \infty$ and consider $\mathcal{F}$ the family of pairs $(f,g) = \big((\mathcal{A}F)^{2\, s_0}, \alpha^{-n} (\mathcal{A}^\alpha F)^{2\, s_0}\big)$. Notice that (2.51) immediately gives that, for every $w \in RH_{s_0'}$,

$$\int_{\mathbb{R}^n} f(x)^{\frac{1}{s_0}}\, w(x)\, dx = \int_{\mathbb{R}^n} \mathcal{A}F(x)^2\, w(x)\, dx \le C\, \alpha^{-\frac{n}{s_0}} \int_{\mathbb{R}^n} \mathcal{A}^\alpha F(x)^2\, w(x)\, dx = C \int_{\mathbb{R}^n} g(x)^{\frac{1}{s_0}}\, w(x)\, dx,$$

where C does not depend on α. Next, we apply (*b*) in Theorem 1.46 to conclude that, for every $1 < s < \infty$ and for every $w \in RH_{s'}$,

$$\int_{\mathbb{R}^n} \mathcal{A}F(x)^{\frac{2\, s_0}{s}}\, w(x)\, dx = \int_{\mathbb{R}^n} f(x)^{\frac{1}{s}}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{\frac{1}{s}}\, w(x)\, dx = C\, \alpha^{-\frac{n}{s}} \int_{\mathbb{R}^n} \mathcal{A}^\alpha F(x)^{\frac{2\, s_0}{s}}\, w(x)\, dx,$$

where C does not depend on α. From this, using that $1 \le s_0 < \infty$ is arbitrary we conclude (2.50) under the restriction $1 < s < \infty$.

Finally, we show (2.50) for all $2 < p < \infty$ and $w \in RH_\infty$ (i.e., $s = 1$). Without loss of generality, we may assume that $\alpha > 32$ (for $1 \le \alpha \le 32$ we just use that $\mathcal{A}F \le \mathcal{A}^\alpha F$). Let us also assume that $\|\mathcal{A}^\alpha F\|_{L^p(w)} < \infty$. Otherwise, there is nothing to prove. Besides, since $w \in RH_\infty$ there exists $r > 1$, which can be assumed to satisfy $r \ge p/2$, such that $w \in A_r$. Then we can apply part (*i*) with $\beta = 6\sqrt{n}\alpha$ and obtain that

$$\|\mathcal{A}^{6\sqrt{n}\alpha} F\|_{L^p(w)} \le C \left(\frac{6\sqrt{n}\alpha}{\alpha} \right)^{\frac{n r}{p}} \|\mathcal{A}^\alpha F\|_{L^p(w)} = C \|\mathcal{A}^\alpha F\|_{L^p(w)} < \infty, \tag{2.52}$$

where C does not depend on α.

After these observations, for every $\lambda > 0$, consider the set

$$O_\lambda := \{x \in \mathbb{R}^n : \mathcal{A}^{6\sqrt{n}\alpha} F(x) > \lambda\}.$$

We shall show that

$$w(\{x \in \mathbb{R}^n : \mathcal{A}F(x) > 2\lambda\}) \lesssim \frac{\alpha^{-n}}{\lambda^2} \int_{O_\lambda} |\mathcal{A}^{6\sqrt{n}\alpha} F(x)|^2 w(x) dx. \tag{2.53}$$

Note that the previous estimate is trivial when $O_\lambda = \emptyset$: both sides vanish since $\mathcal{A}F \le \mathcal{A}^{6\sqrt{n}\alpha} F$. We may then assume that $O_\lambda \neq \emptyset$. From the arguments in the proof of (i) we clearly have that O_λ is open. Also (2.52) and Chebychev's inequality give that $w(O_\lambda) < \infty$, which in turn yields that $O_\lambda \subsetneq \mathbb{R}^n$. We can then take a Whitney decomposition of O_λ (cf. [75, Chapter VI]): there exists a family of closed cubes $\{Q_j\}_{j\in\mathbb{N}}$ with disjoint interiors so that

$$O_\lambda = \bigcup_{j\in\mathbb{N}} Q_j, \qquad \operatorname{diam}(Q_j) \le d(Q_j, \mathbb{R}^n \setminus O_\lambda) \le 4\operatorname{diam}(Q_j), \qquad \sum_j \mathbf{1}_{Q_j^*} \le 12^n\, \mathbf{1}_{O_\lambda}, \tag{2.54}$$

where $Q_j^* := \frac{9}{8} Q_j$.

On the other hand, since $\mathcal{A}F \le \mathcal{A}^{6\sqrt{n}\alpha} F$, we have that

$$w(\{x \in \mathbb{R}^n : \mathcal{A}F(x) > 2\lambda\}) = w(\{x \in O_\lambda : \mathcal{A}F(x) > 2\lambda\}) = \sum_{j\in\mathbb{N}} w(\{x \in Q_j : \mathcal{A}F(x) > 2\lambda\}). \tag{2.55}$$

Fix $j \in \mathbb{N}$ and, for every $x \in Q_j$, write

$$\mathcal{A}F(x) \le G_j(x) + H_j(x) := \left(\int_{\frac{\ell(Q_j)}{\alpha}}^{\infty} \int_{B(x,t)} |F(y,t)|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}} + \left(\int_0^{\frac{\ell(Q_j)}{\alpha}} \int_{B(x,t)} |F(y,t)|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Pick $x_j \in \mathbb{R}^n \setminus O_\lambda$ such that $d(x_j, Q_j) \le 4\operatorname{diam}(Q_j)$. Notice that for every $x \in Q_j$ and $t \ge \ell(Q_j)/\alpha$ we have that $B(x,t) \subset B(x_j, 6\sqrt{n}\alpha t)$. Then,

$$G_j(x)^2 = \int_{\frac{\ell(Q_j)}{\alpha}}^{\infty} \int_{B(x,t)} |F(y,t)|^2 \frac{dy\, dt}{t^{n+1}} \le \int_{\frac{\ell(Q_j)}{\alpha}}^{\infty} \int_{B(x_j, 6\sqrt{n}\alpha t)} |F(y,t)|^2 \frac{dy\, dt}{t^{n+1}} \le \mathcal{A}^{6\sqrt{n}\alpha} F(x_j)^2 \le \lambda^2,$$

where we have used that $x_j \in \mathbb{R}^n \setminus O_\lambda$ in the last inequality. Using this and that $w \in RH_\infty$, we have

$$\begin{aligned}
w(\{x \in Q_j : \mathcal{A}F(x) > 2\lambda\}) &\le w(\{x \in Q_j : H_j(x) > \lambda\}) \\
&\le \frac{1}{\lambda^2} \int_{Q_j} H_j(x)^2 w(x) dx \\
&\le \frac{1}{\lambda^2} \iint_{\mathcal{R}(Q_j)} \mathbf{1}_{(0, \alpha^{-1}\ell(Q_j))}(t) |F(y,t)|^2 w(B(y,t)) \frac{dy\, dt}{t^{n+1}} \\
&\lesssim \frac{\alpha^{-n}}{\lambda^2} \iint_{\mathcal{R}(Q_j)} \mathbf{1}_{(0, \alpha^{-1}\ell(Q_j))}(t) |F(y,t)|^2 w(B(y, 32^{-1}\alpha t)) \frac{dy\, dt}{t^{n+1}} \\
&\le \frac{\alpha^{-n}}{\lambda^2} \int_{Q_j^*} \int_0^\infty \int_{B(x, 32^{-1}\alpha t)} |F(y,t)|^2 \frac{dy\, dt}{t^{n+1}} w(x) dx \\
&\le \frac{\alpha^{-n}}{\lambda^2} \int_{Q_j^*} \mathcal{A}^{6\sqrt{n}\alpha} F(x)^2 w(x) dx.
\end{aligned}$$

Then, by (2.55) and the bounded overlap of the family $\{Q_j^*\}_{j\in\mathbb{N}}$, we conclude (2.53):

$$w(\{x \in \mathbb{R}^n : \mathcal{A}F(x) > 2\lambda\}) \lesssim \frac{\alpha^{-n}}{\lambda^2} \sum_{j\in\mathbb{N}} \int_{Q_j^*} |\mathcal{A}^{6\sqrt{n}\alpha} F(x)|^2 w(x)dx \lesssim \frac{\alpha^{-n}}{\lambda^2} \int_{O_\lambda} |\mathcal{A}^{6\sqrt{n}\alpha} F(x)|^2 w(x)dx.$$

This, the fact that $2 < p < \infty$, and (2.52) give

$$\begin{aligned}\|\mathcal{A}F\|_{L^p(w)}^p &= 2^p \int_0^\infty p\,\lambda^p\, w\left(\{x : \mathcal{A}F(x) > 2\lambda\}\right) \frac{d\lambda}{\lambda} \lesssim \alpha^{-n} \int_0^\infty \lambda^{p-2} \int_{O_\lambda} \mathcal{A}^{6\sqrt{n}\alpha} F(x)^2 w(x)dx \frac{d\lambda}{\lambda} \\ &\lesssim \alpha^{-n} \int_{\mathbb{R}^n} \mathcal{A}^{6\sqrt{n}\alpha} F(x)^2 \int_0^{\mathcal{A}^{6\sqrt{n}\alpha}F(x)} \lambda^{p-2} \frac{d\lambda}{\lambda} w(x)dx \lesssim \alpha^{-n} \|\mathcal{A}^{6\sqrt{n}\alpha} F\|_{L^p(w)}^p \lesssim \alpha^{-n} \|\mathcal{A}^\alpha F\|_{L^p(w)}^p.\end{aligned}$$

This completes the proof. □

As announced before, we next discuss the sharpness of Proposition 2.43.

Remark 2.56. *Let us consider the weights $w_\theta(x) = |x|^{-\theta}$. It is standard to show that $w_\theta \in A_{\widehat{r}}$ if and only if $-n(\widehat{r} - 1) < \theta < n$ (with the possibility of taking $\theta = 0$ when $\widehat{r} = 1$). Besides, $w_\theta \in RH_{s'}$ if and only if $-\infty < \theta < \frac{n}{s'}$ (with the possibility of taking $\theta = 0$ when $s = 1$). We shall use this family of weights to show that the exponents obtained in Proposition 2.43 parts (i) and (ii) are sharp.*

We proceed as in [4], where the unweighted case was considered. We define $B := B\left(0, \frac{1}{4}\right)$ and $a(y,t) := \mathbf{1}_B(y)\mathbf{1}_{[\frac{1}{2},1]}(t)$. It is straightforward to show that

$$\mathcal{A}a(x) \le C\mathbf{1}_{5B}(x), \quad \forall\, x \in \mathbb{R}^n, \quad \textit{and} \quad \mathcal{A}a(x) \ge C, \quad \forall\, x \in B,$$

and, for every $\alpha \ge 1$,

$$\mathcal{A}^\alpha a(x) \le C\mathbf{1}_{(4\alpha+1)B}(x), \quad \forall\, x \in \mathbb{R}^n, \quad \textit{and} \quad \mathcal{A}^\alpha a(x) \ge C, \quad \forall\, x \in (2\alpha - 1)B.$$

Hence,

$$\|\mathcal{A}a\|_{L^p(w_\theta)} \approx 1 \quad \textit{and} \quad \|\mathcal{A}^\alpha a\|_{L^p(w_\theta)} \approx \alpha^{\frac{n-\theta}{p}}, \tag{2.57}$$

where the implicit constants may depend on θ but are independent of α.

To see that the exponent in part (i) is sharp, assume by way of contradiction, that there exists $0 < \varrho < \frac{n\widehat{r}}{p}$ such that for all $\alpha \ge 1$, $w \in A_{\widehat{r}}$, $1 \le \widehat{r} < \infty$, and $0 < p \le 2\widehat{r}$ there holds

$$\|\mathcal{A}^\alpha F\|_{L^p(w)} \le C_w \alpha^{\frac{n\widehat{r}}{p} - \varrho} \|\mathcal{A}F\|_{L^p(w)}. \tag{2.58}$$

Take $1 \le \widehat{r} < \infty$ $0 < p \le 2\widehat{r}$, and set $\theta := -n(\widehat{r} - 1) + \frac{\varrho p}{2}$. Note that $-n(\widehat{r} - 1) < \theta < n$ and therefore $w_\theta \in A_{\widehat{r}}$. Applying (2.57) and (2.58), there exists C_θ so that for every $\alpha > 1$ there holds

$$\alpha^{\frac{n\widehat{r}}{p} - \frac{\varrho}{2}} = \alpha^{\frac{n-\theta}{p}} \approx \|\mathcal{A}^\alpha a\|_{L^p(w_\theta)} \le C_\theta \alpha^{\frac{n\widehat{r}}{p} - \varrho} \|\mathcal{A}a\|_{L^p(w_\theta)} \approx C_\theta\, \alpha^{\frac{n\widehat{r}}{p} - \varrho},$$

where the implicit constants may depend on θ but are independent of α. This clearly leads to a contradiction since $\alpha^{\frac{n\widehat{r}}{p} - \frac{\varrho}{2}} \gg \alpha^{\frac{n\widehat{r}}{p} - \varrho}$ when $\alpha \to \infty$.

We next see that the exponent in part (ii) is sharp. Again we proceed by way of contradiction: let us assume that there exists $\varrho > 0$ such that for all $\alpha \ge 1$, $w \in RH_{s'}$, $1 \le s < \infty$, and $\frac{2}{s} \le p < \infty$ there holds

$$\|\mathcal{A}F\|_{L^p(w)} \le C_w \alpha^{-\frac{n}{sp} - \varrho} \|\mathcal{A}^\alpha F\|_{L^p(w)}. \tag{2.59}$$

Take $1 \le s < \infty$, $\frac{2}{s} \le p < \infty$, and pick $\theta := \frac{n}{s'} - \frac{\varrho p}{2}$. Observe that $-\infty < \theta < \frac{n}{s'}$ and therefore $w_\theta \in RH_{s'}$. Applying (2.57) and (2.59), there exists C_θ so that for every $\alpha > 1$ there holds

$$1 \approx \|\mathcal{A}a\|_{L^p(w_\theta)} \le C_\theta \alpha^{-\frac{n}{sp} - \varrho} \|\mathcal{A}^\alpha a\|_{L^p(w_\theta)} \approx C_\theta\, \alpha^{-\frac{n}{sp} - \varrho + \frac{n-\theta}{p}} = C_\theta\, \alpha^{-\frac{\varrho}{2}},$$

where the implicit constants may depend on θ but are independent of α. Note that the right-hand side tends to 0 as $\alpha \to \infty$ and this readily leads to a contradiction.

Proposition 2.43 gives us a way to compare the norms of $\mathcal{A}^\alpha F$ in $L^p(w)$ for different angles α. In that result, the emphasis is on the class of weights: fixed a class of weights ($A_{\widehat{r}}$ in (a) or $RH_{s'}$ in (b)), we estimate the change of angles in $L^p(w)$ for some range of p's. In some other situations it may be interesting to give formulas where the emphasis is on the exponent p. This is contained in the following result whose elementary proof follows from Proposition 2.43 and is left to the interested reader:

Proposition 2.60. *Let $w \in A_\infty$, $0 < \alpha \le \beta < \infty$ and $0 < p < \infty$. There hold:*

(i) $\|\mathcal{A}^\beta F\|_{L^p(w)} \le C \left(\dfrac{\beta}{\alpha}\right)^{\frac{n\widehat{r}}{p}} \|\mathcal{A}^\alpha F\|_{L^p(w)}$, *for* $\widehat{r} > \max\{\frac{p}{2}, r_w\}$, *and for* $\widehat{r} = \max\{\frac{p}{2}, r_w\}$ *if* $r_w < \frac{p}{2}$ *or* $w \in A_1$.

(ii) $\|\mathcal{A}^\alpha F\|_{L^p(w)} \le C \left(\dfrac{\alpha}{\beta}\right)^{\frac{n}{sp}} \|\mathcal{A}^\beta F\|_{L^p(w)}$, *for* $\frac{1}{s} < \min\{\frac{p}{2}, \frac{1}{s_w}\}$, *and for* $\frac{1}{s} = \min\{\frac{p}{2}, \frac{1}{s_w}\}$ *if* $\frac{p}{2} < \frac{1}{s_w}$ *or* $w \in RH_\infty$.

Related to change of angles, we establish the following results, which, even though they are not related to tent spaces, they will be required in the proof of some results in Chapter 3.

Proposition 2.61. *Let $1 \le q \le s < \infty$, $w \in RH_{s'}$, and $0 \le \alpha \le 1$. Then, for every $t > 0$, we have*

$$\int_{\mathbb{R}^n} \left(\int_{B(x,\alpha t)} |h(y,t)|\, dy \right)^{\frac{1}{q}} w(x)dx \lesssim \alpha^{\frac{n}{s}} \int_{\mathbb{R}^n} \left(\int_{B(x,t)} |h(y,t)|\, dy \right)^{\frac{1}{q}} w(x)dx. \tag{2.62}$$

Proof. We fix $t > 0$, $0 < \alpha \le 1$, and $1 \le q < \infty$. Set

$$G^\alpha(x,t) := \left(\int_{B(x,\alpha t)} |h(y,t)|\, dy \right)^{\frac{1}{q}}.$$

For $\alpha = 1$, we simply write $G(x,t)$. Then, from (1.12), for all $1 \le s_0 < \infty$ and $w \in RH_{s_0'}$, we have

$$\int_{\mathbb{R}^n} G^\alpha(x,t)^q w(x)dx = \int_{\mathbb{R}^n} |h(y,t)|\, w(B(y,\alpha t))\, dy \lesssim \alpha^{\frac{n}{s_0}} \int_{\mathbb{R}^n} |h(y,t)|\, w(B(y,t))\, dy = \alpha^{\frac{n}{s_0}} \int_{\mathbb{R}^n} G(x,t)^q\, w(x)dx. \tag{2.63}$$

This gives (2.62) for $q = 1$, and thus we may assume that $q > 1$. We shall extrapolate from (2.63). Take an arbitrary $1 \le s_0 < \infty$ and consider $\mathcal{F}$ the family of pairs $(f,g) = \big(G^\alpha(\cdot,t)^{q\,s_0}, \alpha^n\, G(\cdot,t)^{q\,s_0}\big)$. Notice that (2.63) immediately gives that, for every $w \in RH_{s_0'}$,

$$\int_{\mathbb{R}^n} f(x)^{\frac{1}{s_0}}\, w(x)\, dx = \int_{\mathbb{R}^n} G^\alpha(x,t)^q\, w(x)\, dx \le C\, \alpha^{\frac{n}{s_0}} \int_{\mathbb{R}^n} G(x,t)^q\, w(x)\, dx = C \int_{\mathbb{R}^n} g(x)^{\frac{1}{s_0}}\, w(x)\, dx,$$

where C does not depend on α. Next, we apply (b) in Theorem 1.46 to conclude that, for every $1 < s < \infty$ and for every $w \in RH_{s'}$,

$$\int_{\mathbb{R}^n} G^\alpha(x,t)^{\frac{q\,s_0}{s}}\, w(x)\, dx = \int_{\mathbb{R}^n} f(x)^{\frac{1}{s}}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{\frac{1}{s}}\, w(x)\, dx = C\, \alpha^{\frac{n}{s}} \int_{\mathbb{R}^n} G(x,t)^{\frac{q\,s_0}{s}}\, w(x)\, dx,$$

where C does not depend on α. From this, if $1 < q \le s < \infty$ we can take $s_0 = s/q$ and conclude (2.62) as desired. $\square$

If we now impose some restriction on the parameter t, we can keep some control over the support of the integral on x.

Proposition 2.64. *Let $1 \leq q \leq s < \infty$, $w \in RH_{s'}$, and $0 \leq \alpha \leq 1$. Then, for every ball B with radius r_B, and $0 < t \leq r_B$, there hold, for $j \geq 2$,*

$$\int_{C_j(B)} \left(\int_{B(x,\alpha t)} |h(y,t)|\, dy \right)^{\frac{1}{q}} w(x)dx \lesssim \alpha^{\frac{n}{s}} \int_{2^{j+3}B \setminus 2^{j-2}B} \left(\int_{B(x,t)} |h(y,t)|\, dy \right)^{\frac{1}{q}} w(x)dx; \tag{2.65}$$

and,

$$\int_{4B} \left(\int_{B(x,\alpha t)} |h(y,t)|\, dy \right)^{\frac{1}{q}} w(x)dx \lesssim \alpha^{\frac{n}{s}} \int_{6B} \left(\int_{B(x,t)} |h(y,t)|\, dy \right)^{\frac{1}{q}} w(x)dx. \tag{2.66}$$

Proof. We fix $0 < t \leq r_B$, $0 < \alpha \leq 1$, and $1 \leq q < \infty$, and set

$$G^{\alpha}(x,t) := \left(\int_{B(x,\alpha t)} |h(y,t)|\, dy \right)^{\frac{1}{q}}.$$

For $\alpha = 1$, we simply write $G(x,t)$. Note that, for all $x \in C_j(B)$, $0 < t \leq r_B$, and $0 < \alpha \leq 1$, we have that $B(x,\alpha t) \subset 2^{j+2}B \setminus 2^{j-1}B$, for all $j \geq 2$, and $B(x,\alpha t) \subset 5B$, for $j = 1$. Besides, if $y \in 2^{j+2}B \setminus 2^{j-1}B$ and $0 < t \leq r_B$, then $B(y,t) \subset 2^{j+3}B \setminus 2^{j-2}B$, for all $j \geq 2$; on the other hand if $y \in 5B$ and $0 < t \leq r_B$, then $B(y,t) \subset 6B$, for $j = 1$.

Hence, for $\mathcal{E}_1 = 2^{j+2}B \setminus 2^{j-1}B$ or $5B$, if $j \geq 2$ or $j = 1$, respectively; and $\mathcal{E}_2 = 2^{j+3}B \setminus 2^{j-2}B$ or $6B$, if $j \geq 2$ or $j = 1$, respectively, from (1.12), for all $1 \leq s_0 < \infty$ and $w \in RH_{s_0'}$, we have

$$\int_{C_j(B)} G^{\alpha}(x,t)^q w(x)dx \leq \int_{\mathcal{E}_1} |h(y,t)|\, w(B(y,\alpha t))\, dy \lesssim \alpha^{\frac{n}{s_0}} \int_{\mathcal{E}_1} |h(y,t)|\, w(B(y,t))\, dy \leq \alpha^{\frac{n}{s_0}} \int_{\mathcal{E}_2} G(x,t)^q\, w(x)dx. \tag{2.67}$$

This gives (2.65) for $q = 1$, and thus we may assume that $q > 1$. We shall extrapolate from (2.67). Take an arbitrary $1 \leq s_0 < \infty$ and consider $\mathcal{F}$ the family of pairs $(f,g) = \left(\mathbf{1}_{C_j(B)} G^{\alpha}(\cdot,t)^{q\, s_0}, \alpha^n\, \mathbf{1}_{\mathcal{E}_2} G(\cdot,t)^{q\, s_0} \right)$. Notice that (2.67) immediately gives that, for every $w \in RH_{s_0'}$,

$$\int_{\mathbb{R}^n} f(x)^{\frac{1}{s_0}}\, w(x)\, dx = \int_{C_j(B)} G^{\alpha}(x,t)^q\, w(x)\, dx \leq C\, \alpha^{\frac{n}{s_0}} \int_{\mathcal{E}_2} G(x,t)^q\, w(x)\, dx = C \int_{\mathbb{R}^n} g(x)^{\frac{1}{s_0}}\, w(x)\, dx,$$

where C does not depend on α. Next, we apply (b) in Theorem 1.46 to conclude that, for every $1 < s < \infty$ and for every $w \in RH_{s'}$,

$$\int_{C_j(B)} G^{\alpha}(x,t)^{\frac{q\, s_0}{s}}\, w(x)\, dx = \int_{\mathbb{R}^n} f(x)^{\frac{1}{s}}\, w(x)\, dx \leq C \int_{\mathbb{R}^n} g(x)^{\frac{1}{s}}\, w(x)\, dx = C\, \alpha^{\frac{n}{s}} \int_{\mathcal{E}_2} G(x,t)^{\frac{q\, s_0}{s}}\, w(x)\, dx,$$

where C does not depend on α. From this, if $1 < q \leq s < \infty$ we can take $s_0 = s/q$ and conclude (2.65) and (2.66) as desired. □

2.3.2 Comparability of the operators $\mathcal{A}_r$ and $C_{r,p}$

Let us recall the definition of the operator $\widehat{C}_r$ in Section 2.1:

$$\widehat{C}_r F(x) := \sup_{B \ni x} \left(\frac{1}{|B|} \int_0^{r_B} \int_B |F(y,t)|^r \frac{dy\, dt}{t} \right)^{\frac{1}{r}},$$

where we just write $\widehat{C}$ when $r = 2$.

Given $0 < p, r < \infty$, we now introduce a new maximal operator (which has been used for instance in [60]):

$$C_{r,p}F(x_0) = \sup_{B\ni x_0} \left(\frac{1}{|B|} \int_B \left(\int_0^{r_B} \int_{B(x,t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p}{r}} dx \right)^{\frac{1}{p}}, \tag{2.68}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and where r_B denotes the corresponding radius. We just write C_p when $r = 2$.

This operator is a version of $\widehat{C}_r$ which will be very useful for our purposes. Indeed, for $p = r$, we shall see that $\widehat{C}_r F \approx C_{r,r}F$. First, applying Fubini we have

$$C_{r,r}F(x_0) = \sup_{B\ni x_0} \left(\frac{1}{|B|} \int_B \int_0^{r_B} \int_{B(x,t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\, dx \right)^{\frac{1}{r}} \leq \sup_{B\ni x_0} \left(\frac{1}{|B|} \int_{2B} \int_0^{r_B} |F(y,t)|^r \int_{B(y,t)} 1\, dx \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{r}}$$
$$\lesssim \sup_{B\ni x_0} \left(\frac{1}{|2B|} \int_0^{2r_B} \int_{2B} |F(y,t)|^r \frac{dy\,dt}{t} \right)^{\frac{1}{r}} = \widehat{C}_r F(x_0).$$

For the reverse inequality, there holds

$$\widehat{C}_r F(x_0) = \sup_{B\ni x_0} \left(\frac{1}{|B|} \int_0^{r_B} \int_B |F(y,t)|^r \frac{dy\,dt}{t} \right)^{\frac{1}{r}} \lesssim \sup_{B\ni x_0} \left(\frac{1}{|B|} \int_0^{r_B} \int_B |F(y,t)|^r \int_{B(y,t)} 1\, dx \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{r}}$$
$$\lesssim \sup_{B\ni x_0} \left(\frac{1}{|2B|} \int_{2B} \int_0^{2r_B} \int_{B(x,t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\, dx \right)^{\frac{1}{r}} = C_{r,r}F(x_0).$$

We have a weighted result similar to that of Theorem 2.7, part (*ii*), where it is stated that $\mathcal{A}_r F$ and $\widehat{C}_r$ are comparable in $L^p(\mathbb{R}^n)$ for every $< p < \infty$. Our result gives comparability of $\mathcal{A}_r F$ and $C_{r,p_0}F$ in the range $p_0 < p < \infty$ and, in particular, if $p_0 < r$ we can go below $p = r$. We prove the case $r = 2$ in the next proposition. For a general r, the result is stated in Proposition 2.75 and to prove it, we just need to observe that $C_{r,p_0}F(x) = C_{\frac{2p_0}{r}}\big(|F|^{\frac{r}{2}}\big)(x)^{\frac{2}{r}}$ and apply the case $r = 2$.

Proposition 2.69.

(*a*) *If* $0 < p_0, p < \infty$, $w \in A_\infty$ *and* $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ *then*

$$\|\mathcal{A}F\|_{L^p(w)} \lesssim \|C_{p_0}F\|_{L^p(w)}.$$

(*b*) *If* $0 < p_0 < p < \infty$ *and* $w \in A_{\frac{p}{p_0}}$ *then*

$$\|C_{p_0}F\|_{L^p(w)} \lesssim \|\mathcal{A}F\|_{L^p(w)}.$$

Proof. The proof of (*a*) uses a good-λ argument. Then, it requires to know that the quantity to be hidden is a priori finite. To guarantee this we divide the proof into two steps. The first step consists in proving (*a*) for all $F \in L^2(\mathbb{R}^{n+1}_+)$ such that, for some $N > 1$, $\operatorname{supp} F \subset K_N := \mathbf{1}_{B(0,N)}(y)\mathbf{1}_{(N^{-1},N)}(t)$. In the second step we will consider general functions $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and define, $F_N := F\mathbf{1}_{K_N}$, $N \geq 1$. Clearly $F_N \in L^2(\mathbb{R}^{n+1}_+)$ and $\operatorname{supp} F_N \subset K_N$, and hence we can apply step 1 to F_N. By a limiting argument we will obtain the desired estimate for F.

Step 1: Take $F \in L^2(\mathbb{R}^{n+1}_+)$ such that, for some $N > 1$, $\operatorname{supp} F \subset K_N$, and note that under this assumption $\|\mathcal{A}F\|_{L^p(w)} < \infty$. Indeed, $\operatorname{supp} \mathcal{A}F \subset B(0,2N)$, and then

$$\|\mathcal{A}F\|_{L^p(w)} \leq N^{\frac{n+1}{2}} \|F\|_{L^2(\mathbb{R}^{n+1}_+)} w(B(0,2N))^{\frac{1}{p}} < \infty.$$

We claim that it is enough to prove that there exist $\alpha > 1$ and a constant c such that for all $0 < \gamma \le 1$ and $0 < \lambda < \infty$ we have

$$w(\{x \in \mathbb{R}^n : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}) \le c\gamma^{c_w} w(\{x \in \mathbb{R}^n : \mathcal{A}^\alpha F(x) > \lambda\}). \tag{2.70}$$

Assuming this momentarily it follows that

$$\begin{aligned} w(\{x \in \mathbb{R}^n : \mathcal{A}F(x) > 2\lambda\}) &\le w(\{x \in \mathbb{R}^n : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}) + w(\{x \in \mathbb{R}^n : C_{p_0}F(x) > \gamma\lambda\}) \\ &\le c\gamma^{c_w} w(\{x \in \mathbb{R}^n : \mathcal{A}^\alpha F(x) > \lambda\}) + w(\{x \in \mathbb{R}^n : C_{p_0}F(x) > \gamma\lambda\}). \end{aligned}$$

This easily gives

$$\|\mathcal{A}F\|^p_{L^p(w)} \le C_{\gamma,p}\|C_{p_0}F\|^p_{L^p(w)} + c\gamma^{c_w}\|\mathcal{A}^\alpha F\|^p_{L^p(w)}.$$

From Proposition 2.43 we know that $\|\mathcal{A}^\alpha F\|_{L^p(w)} \le c(\alpha, p)\|\mathcal{A}F\|_{L^p(w)}$. Then, by choosing γ small enough so that $c\gamma^{c_w}c(\alpha, p)^p < 1$, and since $\|\mathcal{A}F\|_{L^p(w)} < \infty$, we easily conclude that

$$\|\mathcal{A}F\|_{L^p(w)} \lesssim \|C_{p_0}F\|_{L^p(w)}.$$

To complete the proof it remains to show (2.70). We argue as in [32]. Write $O_\lambda = \{x \in \mathbb{R}^n : \mathcal{A}^\alpha F(x) > \lambda\}$. We may assume that $w(O_\lambda) < \infty$ (otherwise, there is nothing to prove) and this in turn implies that $O_\lambda \subsetneq \mathbb{R}^n$. Without loss of generality we can also suppose that $O_\lambda \ne \emptyset$ (otherwise, both terms in (2.70) vanish, since $\mathcal{A}^\alpha F \ge \mathcal{A}F$ because $\alpha > 1$, and again the proof is trivial). Note finally that O_λ is open, fact that can be proved much as in the proof of Proposition 2.43. We can then take a Whitney decomposition of O_λ (cf. [75, Chapter VI]): there exists a family of closed cubes $\{Q_j\}_{j\in\mathbb{N}}$ with disjoint interiors satisfying (2.54). In particular, for each $j \in \mathbb{N}$ we can pick $x_j \in \mathbb{R}^n \setminus O_\lambda$ such that $d(x_j, Q_j) \le 4\mathrm{diam}(Q_j)$. Furthermore, since $\alpha > 1$ we have $\mathcal{A}^\alpha F \ge \mathcal{A}F$ and

$$\begin{aligned} w(\{x \in \mathbb{R}^n : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}) &= w(\{x \in O_\lambda : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}) \\ &= \sum_{j\in\mathbb{N}} w(\{x \in Q_j : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}). \end{aligned}$$

Thus, to show (2.70), it is enough to prove

$$|\{x \in Q_j : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}| \le c\gamma^{p_0}|Q_j|, \tag{2.71}$$

which, together with $w \in A_\infty$ (cf. (1.10)), would imply

$$w(\{x \in Q_j : \mathcal{A}F(x) > 2\lambda, C_{p_0}F \le \gamma\lambda\}) \le c\gamma^{c_w} w(Q_j),$$

and summing in j we would get (2.70).

Let us now fix $j \in \mathbb{N}$ and obtain (2.71). There is nothing to prove if the set on its left-hand side is empty. Thus, we assume that there exists $\bar{x}_j \in \{x \in Q_j : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}$. Let B_j be the ball such that $Q_j \subset B_j$ with $2r_{B_j} = \mathrm{diam}(Q_j)$. Then, $d(x_j, Q_j) \le 8r_{B_j}$ and $Q_j \subset \overline{B(x_j, 10r_{B_j})}$.

We now write

$$F(x,t) = F_{1,j}(x,t) + F_{2,j}(x,t) := F(x,t)\,\mathbf{1}_{[r_{B_j},\infty)}(t) + F(x,t)\,\mathbf{1}_{(0,r_{B_j})}(t).$$

In particular, $\mathcal{A}F(x) \le \mathcal{A}F_{1,j}(x) + \mathcal{A}F_{2,j}(x)$. Easy calculations lead to obtain that for every $\alpha \ge 11$ there holds

$$\mathcal{A}F_{1,j}(x)^2 = \int_{r_{B_j}}^\infty \int_{|x-y|<t} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \le \int_0^\infty \int_{|x_j-y|<\alpha t} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} = \mathcal{A}^\alpha F(x_j)^2 \le \lambda^2, \tag{2.72}$$

where in the last inequality we have used the fact that $x_j \in \mathbb{R}^n \setminus O_\lambda$. On the other hand, by our choice of $\bar{x}_j \in Q_j \subset B_j$, it follows that

$$\frac{1}{|B_j|}\int_{B_j} \mathcal{A}F_{2,j}(x)^{p_0}\,dx = \frac{1}{|B_j|}\int_{B_j}\left(\int_0^{r_{B_j}}\int_{B(x,t)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p_0}{2}} dx \le C_{p_0}F(\bar{x}_j)^{p_0} \le (\gamma\lambda)^{p_0}, \tag{2.73}$$

Using (2.72), Chebychev's inequality, and (2.73) we conclude (2.71):

$$\begin{aligned}|\{x \in Q_j : \mathcal{A}F(x) > 2\lambda, C_{p_0}F(x) \le \gamma\lambda\}| &\le |\{x \in Q_j : \mathcal{A}F_{2,j}(x) > \lambda\}| \\ &\le \frac{1}{\lambda^{p_0}}\int_{Q_j} \mathcal{A}F_{2,j}(x)^{p_0}\,dx \le \gamma^{p_0}|B_j| \le c\gamma^{p_0}|Q_j|.\end{aligned}$$

This completes the proof of Step 1.

Step 2: Take $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and define, for every $N > 1$, $F_N := F\mathbf{1}_{K_N}$. Then, since $F_N \in L^2(\mathbb{R}^{n+1}_+)$ and $\operatorname{supp} F_N \subset K_N$, we can apply Step 1 and obtain that

$$\|\mathcal{A}F_N\|_{L^p(w)} \lesssim \|C_{p_0}F_N\|_{L^p(w)} \le \|C_{p_0}F\|_{L^p(w)},$$

where the implicit constant is uniform in N. Finally since $F_N \nearrow F$ in $\mathbb{R}^{n+1}_+$, the Monotone Convergence Theorem yields the desired estimate. This finishes the proof of (a).

We next turn to prove (b). For every $x_0 \in \mathbb{R}^n$ and any ball $B \subset \mathbb{R}^n$ such that $x_0 \in B$, we have

$$\left(\fint_B \left(\int_0^{r_B}\int_{B(x,t)} |F(y,t)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p_0}{2}} dx\right)^{\frac{1}{p_0}} \le \left(\fint_B |\mathcal{A}F(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}} \le \mathcal{M}_{p_0}(\mathcal{A}F)(x_0),$$

where for any function, h, $\mathcal{M}_{p_0}h(x) := \mathcal{M}\big(|h|^{p_0}\big)(x)^{1/p_0}$. Taking the supremum over all balls containing x_0, we conclude that $C_{p_0}F(x_0) \le \mathcal{M}_{p_0}(\mathcal{A}F)(x_0)$. Besides, since $\mathcal{M}_{p_0} : L^p(w) \to L^p(w)$ (because $w \in A_{\frac{p}{p_0}}$ and $p > p_0$) we finally conclude that

$$\|C_{p_0}F\|_{L^p(w)} \le \|\mathcal{M}_{p_0}(\mathcal{A}F)\|_{L^p(w)} \lesssim \|\mathcal{A}F\|_{L^p(w)}.$$

This completes the proof. □

As we said above, from the case $r = 2$ we can obtain analogous results of Propositions 2.43 and 2.69, for a general $0 < r < \infty$.

Proposition 2.74. *Let* $0 < r < \infty$ *and* $0 < \alpha \le \beta < \infty$.

(i) *For every* $w \in A_q$, $1 \le q < \infty$, *there holds*

$$\|\mathcal{A}_r^\beta F\|_{L^p(w)} \le C\left(\frac{\beta}{\alpha}\right)^{\frac{nq}{p}} \|\mathcal{A}_r^\alpha F\|_{L^p(w)}, \quad \textit{for all} \quad 0 < p \le rq.$$

(ii) *For every* $w \in RH_{s'}$, $1 \le s < \infty$, *there holds*

$$\|\mathcal{A}_r^\alpha F\|_{L^p(w)} \le C\left(\frac{\alpha}{\beta}\right)^{\frac{n}{sp}} \|\mathcal{A}_r^\beta F\|_{L^p(w)}, \quad \textit{for all} \quad \frac{r}{s} \le p < \infty.$$

Proposition 2.75. (a) *If* $0 < p_0, p < \infty$, $w \in A_\infty$, *and* $F \in L^r_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ *then*

$$\|\mathcal{A}_rF\|_{L^p(w)} \lesssim \|C_{r,p_0}F\|_{L^p(w)}.$$

(b) *If* $0 < p_0 < p < \infty$ *and* $w \in A_{\frac{p}{p_0}}$ *then*

$$\|C_{r,p_0}F\|_{L^p(w)} \lesssim \|\mathcal{A}_rF\|_{L^p(w)}.$$

2.3.3 Complex interpolation

We now give a complex interpolation result analogous to that of Theorem 2.7, part (*iv*) in the unweighted case (a real interpolation result in weighted tent spaces is also proved in [27]). The result is the following:

Theorem 2.76. *Suppose* $1 \le p_0 < p_1 < \infty$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$. *Then*

$$[T^{p_0}(w), T^{p_1}(w)]_\theta = T^p(w).$$

Remark 2.77. *Note that in our definition of weighted tent spaces the operator $\mathcal{A}$ is the same operator as in the unweighted case, i.e., it does not depend on the weight w. Consequently, we can not see $\mathbb{R}^n$ with a doubling measure given by a weight w and apply the interpolation results obtained for tent spaces defined in metric measure spaces or spaces of homogeneous type (X, μ, d), since in the definitions of those spaces the operator $\mathcal{A}$ is modified to depend on the measure μ. See for instance [1], [53, Lemma 4.6, Proposition 4.9], or [74].*

Remark 2.78. *In the proof of the inclusion $T^p \subset [T^{p_0}, T^{p_1}]_\theta$, in [32, Lemma 5, Section 7] (following the notation there) the authors claim that the support of the function $\mathcal{A}(f_k)$ is contained in $O_k^* \setminus O_{k+1}$. It is easy to see that* $\operatorname{supp}\mathcal{A}(f_k) \subset O_k^*$, *but it is not clear that the support of $\mathcal{A}(f_k)$ is away from O_{k+1}. In fact, we can construct 1-dimensional examples which show that this is false in general. This was noticed by A. Amenta in [2, Remark 3.20].*

We refer to [20] and [51] for a different proof of that inclusion and hence, of Theorem 2.7, part (iv).

Proof of Theorem 2.76. In order to prove this theorem we just follow the proof of [32, Lemma 4 , Section 7] to show $[T^{p_0}(w), T^{p_1}(w)]_\theta \subset T^p(w)$, and [32, Lemma 5 , Section 7] to show $T^p(w) \subset [T^{p_0}(w), T^{p_1}(w)]_\theta$. However, in view of Remark 2.78, in order to show this last inclusion, we need to complete and slightly modify the proof given in [32, Lemma 5, Section 7].

As we said a few lines above, following the proof of [32, Lemma 4, Section 7], but using interpolation between $L^p(w)$ spaces (note that the proof in [19, Theorem 5.1.1] also works in the weighted case) instead of the usual interpolation in $L^p(\mathbb{R}^n)$ spaces, we get

$$[T^{p_0}(w), T^{p_1}(w)]_\theta \subset T^p(w), \quad 1 \le p_0 < p < p_1 < \infty, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1.$$

As for the converse inclusion, fix $1 \le p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ Take a function $f \in T^p(w)$ such that $\|f\|_{T^p(w)} \le 1$. Then, to see that $f \in [T^{p_0}(w), T^{p_1}(w)]_\theta$ (see Appendix B for the definition of the space $[T^{p_0}(w), T^{p_1}(w)]_\theta$), we need to find a function F, $z \to F(z)$, from the closed strip $0 \le \mathrm{Re}(z) \le 1$ to the Banach space $T^{p_0}(w) + T^{p_1}(w)$ (see also Appendix B for the definition of this space). The function F must be continuous and bounded on the full strip, with respect to the norm of $T^{p_0}(w)+T^{p_1}(w)$, and analytic on the open strip, and such that $F(iy) \in T^{p_0}(w)$ is continuous in $T^{p_0}(w)$ and tends to zero as $|y| \to \infty$, and $F(1+iy) \in T^{p_1}(w)$ is continuous in $T^{p_1}(w)$ and tends to zero as $|y| \to \infty$. Besides, F must satisfy that $F(\theta) = f$ in $T^p(w)$, and

$$\|F(iy)\|_{L^{p_0}(w)} + \|F(1+iy)\|_{L^{p_1}(w)} \le C,$$

uniformly on f. To this end fix $\alpha > 1$ (to be determined during the proof), and, for each $k \in \mathbb{Z}$, consider the sets

$$O_k := \{x \in \mathbb{R}^n : \mathcal{A}^\alpha f(x) > 2^k\}, \quad E_k := \mathbb{R}^n \setminus O_k,$$

and, for a fixed γ, $0 < \gamma < 1$, the set of γ-density

$$E_k^* := \left\{ x \in \mathbb{R}^n : \forall\, r > 0,\ \frac{|E_k \cap B(x,r)|}{|B(x,r)|} \ge \gamma \right\}$$

and its complementary

$$O_k^* := \mathbb{R}^n \setminus E_k^* = \{x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_k})(x) > 1 - \gamma\}.$$

Note that $\operatorname{supp} f \subset \left(\bigcup_{k\in\mathbb{Z}} \widehat{O_k^*} \setminus \widehat{O_{k+1}^*}\right) \bigcup \mathbb{F}$, where $\mathbb{F} \subset \mathbb{R}^{n+1}_+$ and $\int_0^\infty \int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{F}}(y,t)\frac{dy\,dt}{t^{n+1}} = 0$. To see this, observe that

$$\mathbb{R}^{n+1}_+ = \left(\bigcup_{k\in\mathbb{Z}} \widehat{O_k^*} \setminus \widehat{O_{k+1}^*}\right) \bigcup \left(\mathbb{R}^{n+1}_+ \setminus \left(\bigcup_{k\in\mathbb{Z}} \widehat{O_k^*} \setminus \widehat{O_{k+1}^*}\right)\right)$$
$$= \left(\bigcup_{k\in\mathbb{Z}} \widehat{O_k^*} \setminus \widehat{O_{k+1}^*}\right) \bigcup \left(\bigcap_{k\in\mathbb{Z}} \widehat{O_k^*}\right) \bigcup \left(\mathbb{R}^{n+1}_+ \setminus \bigcup_{k\in\mathbb{Z}} \widehat{O_k^*}\right). \quad (2.79)$$

Now, note that, for $w \in A_\infty$ if $r > r_w$ then $\mathcal{M} : L^r(w) \to L^{r,\infty}(w)$ (see Proposition 1.13, part (*vii*)). Hence, applying Chebychev's inequality and Proposition 2.43, we have

$$w(O_k^*) \le c_{\gamma,r} w(O_k) \lesssim \frac{\|\mathcal{A}^\alpha f\|^p_{L^p(w)}}{2^{kp}} \lesssim \frac{\|\mathcal{A} f\|^p_{L^p(w)}}{2^{kp}} < \infty. \quad (2.80)$$

Using this and the fact that $O^*_{k+1} \subset O^*_k$ for all $k \in \mathbb{Z}$, we conclude that

$$w(\cap_{k\in\mathbb{Z}} O_k^*) = \lim_{k\to\infty} w(O_k^*) \lesssim \lim_{k\to\infty} \frac{1}{2^{kp}} = 0,$$

which implies that $|\cap_{k\in\mathbb{Z}} O_k^*| = 0$, since the Lebesgue measure and the measure given by the weight, w, are mutually absolutely continuous. Consequently,

$$\int_0^\infty \int_{\mathbb{R}^n} \mathbf{1}_{\cap_{k\in\mathbb{Z}} \widehat{O_k^*}}(y,t)\frac{dy\,dt}{t^{n+1}} \le \lim_{N\to\infty} \int_{N^{-1}}^N \int_{\mathbb{R}^n} \mathbf{1}_{\cap_{k\in\mathbb{Z}} O_k^*}(y)\frac{dy\,dt}{t^{n+1}} = 0. \quad (2.81)$$

Now observe that $\mathbb{R}^{n+1}_+ \setminus \left(\bigcup_{k\in\mathbb{Z}} \widehat{O_k^*}\right) = \bigcap_{k\in\mathbb{Z}} \mathcal{R}(E_k^*)$, we shall show that

$$f(y,t) = 0 \quad \widetilde{\mu} - \text{a. e. } (y,t) \in \bigcap_{k\in\mathbb{Z}} \mathcal{R}(E_k^*), \quad (2.82)$$

where $\widetilde{\mu}(x,s) := \frac{dx\,ds}{s^{n+1}}$. To his end, consider $\mathbb{F}$ the set of Lebesgue points of $|f(x,s)|^2$ as a function of the variables $(x,s) \in \mathbb{R}^{n+1}_+$ with respect to de measure $dx\,ds$ (note that $\widetilde{\mu}$ and the Lebesgue measure in $\mathbb{R}^{n+1}_+$ are mutually absolutely continuous). Besides, by (2.41) for any compact set $K \subset \mathbb{R}^{n+1}_+$,

$$\iint_K |f(x,s)|^2\, dx\,ds \le C_{K,w} \|\mathcal{A} f\|^2_{L^p(w)} < \infty,$$

then $|f(x,s)|^2 \in L^1_{loc}(\mathbb{R}^{n+1}_+, dx\,ds)$, and hence $\widetilde{\mu}(\mathbb{R}^{n+1}_+ \setminus \mathbb{F}) = 0$. Therefore, in order to conclude (2.82), we just need to prove that

$$f(y,t) = 0,\ \forall (y,t) \in \bigcap_{k\in\mathbb{Z}} \mathcal{R}(E_k^*) \cap \mathbb{F}. \quad (2.83)$$

On the one hand, if $(y,t) \in \bigcap_{k\in\mathbb{Z}} \mathcal{R}(E_k^*)$, for every $k \in \mathbb{Z}$ there exists x_k such that $(y,t) \in \Gamma(x_k)$ and $\mathcal{A}^\alpha f(x_k) \le 2^k$. On the other hand, if $(y,t) \in \mathbb{F}$ we have that,

$$\lim_{r\to 0} \frac{1}{|B((y,t),r)|} \iint_{B((y,t),r)} \left| |f(y,t)|^2 - |f(x,s)|^2 \right| \, dx\,ds = 0. \quad (2.84)$$

Finally, for all $r > 0$, consider

$$x_k^r := \begin{cases} x_k & \text{if} \quad y = x_k, \\ y - \frac{r(y-x_k)}{2|y-x_k|} & \text{if} \quad y \ne x_k. \end{cases}$$

It is easy to see that $B\left((x_k^r,t),\frac{r}{4}\right)\subset\Gamma(x_k)\cap B((y,t),r)\subset\Gamma^{\alpha}(x_k)\cap B((y,t),r)$, for every $k\in\mathbb{Z}$ and $0<r<t$. Combining all these facts we have that, for $(y,t)\in\bigcap_{k\in\mathbb{Z}}\mathcal{R}(E_k^*)\cap\mathbb{F}$,

$$\begin{aligned}
&|f(y,t)|^2\\
&=\frac{1}{|B((x_k^r,t),r/4)|}\iint_{B((x_k^r,t),r/4)}\left||f(y,t)|^2-|f(x,s)|^2\right|\,dx\,ds+\frac{1}{|B((x_k^r,t),r/4)|}\iint_{B((x_k^r,t),r/4)}|f(x,s)|^2dx\,ds\\
&\qquad\lesssim\frac{1}{|B((y,t),r)|}\iint_{B((y,t),r)}\left||f(y,t)|^2-|f(x,s)|^2\right|\,dx\,ds+\frac{(t+r)^{n+1}}{r^{n+1}}4^k.
\end{aligned}$$

Now, taking limits first as $k\to-\infty$ and then as $r\to0$, by (2.84), we conclude (2.83).

Then, by (2.79), (2.81), and (2.82), we can write $f=\sum_{k\in\mathbb{Z}}f\mathbf{1}_{\widehat{O_k^*}\setminus\widehat{O_{k+1}^*}}=:\sum_{k\in\mathbb{Z}}f_k$ in $T^p(w)$.

Finally consider the function

$$F(z):=e^{z^2-\theta^2}\sum_{k\in\mathbb{Z}}2^{k(\alpha(z)p-1)}f_k,$$

where $\alpha(z):=\frac{1-z}{p_0}+\frac{z}{p_1}$. We shall see that F satisfies all the conditions that we mentioned above. First note that $F(\theta)=f$ in $T^p(w)$. Moreover, for all $z\in\mathbb{C}$ such that $0<\mathrm{Re}z<1$, applying Young's inequality, we have that

$$\begin{aligned}
|F(z)|&=\left|e^{z^2-\theta^2}\sum_{k\in\mathbb{Z}}2^{k(\alpha(z)p-1)}f_k\right|\le e\sum_{k\in\mathbb{Z}}2^{k\left(p\frac{1-\mathrm{Re}(z)}{p_0}+p\frac{\mathrm{Re}(z)}{p_1}-1\right)}|f_k|\\
&=e\sum_{k\in\mathbb{Z}}\left(2^{k\left(\frac{p}{p_0}-1\right)}|f_k|\right)^{1-\mathrm{Re}(z)}\left(2^{k\left(\frac{p}{p_1}-1\right)}|f_k|\right)^{\mathrm{Re}(z)}\\
&\le e(1-\mathrm{Re}(z))\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_0}-1\right)}|f_k|+e\mathrm{Re}(z)\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_1}-1\right)}|f_k|\\
&\le e\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_0}-1\right)}|f_k|+e\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_1}-1\right)}|f_k|. \qquad (2.85)
\end{aligned}$$

Besides, for all $-\infty<y<\infty$,

$$|F(iy)|\le e^{-y^2}\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_0}-1\right)}|f_k|\quad\text{and}\quad|F(1+iy)|\le e^{1-y^2}\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_1}-1\right)}|f_k|. \qquad (2.86)$$

Then, in order to see that F satisfies the desired conditions, it suffices to show that

$$\left\|\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_0}-1\right)}|f_k|\right\|_{T^{p_0}(w)}+\left\|\sum_{k\in\mathbb{Z}}2^{k\left(\frac{p}{p_1}-1\right)}|f_k|\right\|_{T^{p_1}(w)}\le C. \qquad (2.87)$$

Indeed, combining this with (2.86), we obtain that $F(iy)\in T^{p_0}(w)$ is continuous in $T^{p_0}(w)$ and tends to zero as $|y|\to\infty$, and $F(1+iy)\in T^{p_1}(w)$ is continuous in $T^{p_1}(w)$ and tends to zero as $|y|\to\infty$. On the other hand, by (2.85), (2.86), and (2.87), we easily obtain that F is a continuous and bounded function with respect to the norm of $T^{p_0}(w)+T^{p_1}(w)$ on the full strip. Finally, to see that F is analytic on the open strip we apply Morera's theorem for Banach-space valued functions. We have that $F(z)$ is continuous, so it just remains to show that for every triangle T in the open set $\widehat{C}:=\{z\in\mathbb{C}:0<\mathrm{Re}z<1\}$, we have that

$$\int_T F(z)=0.$$

To see this, consider for each $k\in\mathbb{Z}$ $g_k(z):=e^{z^2-\theta^2}2^{k(p\alpha(z)-1)}f_k$, we have that these functions are analytic on $\widehat{C}$. Then, for all T triangle in $\widehat{C}$ and each $k\in\mathbb{Z}$, by Cauchy's theorem,

$$\int_T g_k(z)=0.$$

Hence, it suffices to justify that we can take the sum in $k \in \mathbb{Z}$ out of the integral. This follows by the dominated convergence theorem for Bochner integrals. Note that

$$\int_T F(z) = \sum_{j=1}^{3} \int_0^{t_j} F(\gamma_j(t))\gamma_j'(t)dt,$$

where γ_j is a parametrization of each side of the triangle T, and that, by (2.85) and (2.87), for $j = 1, 2, 3$,

$$\int_0^{t_j} \|F(\gamma_j(t))\gamma_j'(t)\|_{T^{p_0}(w)+T^{p_1}(w)}dt \lesssim \int_0^{t_j} \|F(\gamma_j(t))\|_{T^{p_0}(w)+T^{p_1}(w)}dt < \infty.$$

Consequently the function $t \to F(\gamma_j(t))\gamma_j'(t)$, for all $t \in [0, t_j]$ is Bochner integrable. Moreover, for all $M > 0$ and $j = 1, 2, 3$, again by (2.85) and (2.87),

$$\left\| \sum_{|k| \le M} e^{\gamma_j(t)^2 - \theta^2} 2^{k(\alpha(\gamma_j(t))p-1)} f_k \gamma_j'(t) \right\|_{T^{p_0}(w)+T^{p_1}(w)} \lesssim C,$$

which implies that we can apply the dominated convergence theorem for Bochner integrals and then conclude that F is analytic.

Besides, from (2.86), (2.87), and the fact that $F(\theta) = f$ in $T^p(w)$ with $\|f\|_{T^p(w)} \le 1$, we have that $f \in [T^{p_0}(w), T^{p_1}(w)]_\theta$ and that

$$\|f\|_{[T^{p_0}(w),T^{p_1}(w)]_\theta} \lesssim \|f\|_{T^p(w)}.$$

Let us thus prove (2.87). Consider $q = p_0$ or p_1. Then, since $\operatorname{supp}(\mathcal{A}f_k) \subset O_k^*$ and $O_{k+1} \subset O_{k+1}^* \subset O_k^*$, we have

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} 2^{k\left(\frac{p}{q}-1\right)} |f_k| \right\|_{T^q(w)} &\le \sup_{\|\psi\|_{L^{q'}(w)} \le 1} \sum_{k\in\mathbb{Z}} 2^{k\left(\frac{p}{q}-1\right)} \int_{O_k^*} |\mathcal{A}f_k(x)||\psi(x)|w(x)dx \\
&= \sup_{\|\psi\|_{L^{q'}(w)} \le 1} \sum_{k\in\mathbb{Z}} 2^{k\left(\frac{p}{q}-1\right)} \int_{O_k^* \setminus O_{k+1}} |\mathcal{A}f_k(x)||\psi(x)|w(x)dx + \sup_{\|\psi\|_{L^{q'}(w)} \le 1} \sum_{k\in\mathbb{Z}} 2^{k\left(\frac{p}{q}-1\right)} \int_{O_{k+1}} |\mathcal{A}f_k(x)||\psi(x)|w(x)dx \\
&=: \sup_{\|\psi\|_{L^{q'}(w)} \le 1} \sum_{k\in\mathbb{Z}} I_k + \sup_{\|\psi\|_{L^{q'}(w)} \le 1} \sum_{k\in\mathbb{Z}} II_k.
\end{aligned}$$

Using that for $\alpha > 1$, $\mathcal{A}f_k \le \mathcal{A}^\alpha f_k$, and that for $x \in O_k^* \setminus O_{k+1}$, $\mathcal{A}^\alpha f_k(x) \le 2^{k+1}$, we obtain that

$$I_k \lesssim 2^{k\frac{p}{q}} \int_{O_k^*} |\psi(x)|w(x)dx.$$

Next we see that II_k is controlled by the same expression in the right-hand side of the above estimate. For every $k \in \mathbb{Z}$, since $w(O_k^*) \lesssim w(O_k) < \infty$, we can take a Whitney decomposition of O_{k+1}^*: $O_{k+1}^* = \bigcup_{j\in\mathbb{N}} Q_{k+1}^j$, which satisfies

$$\sqrt{n}\ell(Q_{k+1}^j) \le d(Q_{k+1}^j, \mathbb{R}^n \setminus O_{k+1}^*) \le 4\sqrt{n}\ell(Q_{k+1}^j),$$

and the Q_{k+1}^j have disjoint interiors. Now, for every $x \in Q_{k+1}^j$, we split $\mathcal{A}f_k(x)$ as follows

$$\mathcal{A}f_k(x) \le \left(\int_0^{\ell(Q_{k+1}^j)/2} \int_{B(x,t)} |f_k(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}} + \left(\int_{\ell(Q_{k+1}^j)/2}^{\infty} \int_{B(x,t)} |f_k(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}} =: G_1(x) + G_2(x).$$

On one hand, note that

$$\mathcal{E} \bigcap \left(\widehat{O_k^*} \setminus \widehat{O_{k+1}^*} \right) := \left\{ (y,t) \in \Gamma(x) : x \in Q_{k+1}^j, 0 < t < \ell(Q_{k+1}^j)/2 \right\} \bigcap \left(\widehat{O_k^*} \setminus \widehat{O_{k+1}^*} \right) = \emptyset.$$

Indeed for $(y,t) \in \mathcal{E}$, since $d(y, Q^j_{k+1}) \le t$, we have

$$d(y, \mathbb{R}^n \setminus O^*_{k+1}) \ge d(Q^j_{k+1}, \mathbb{R}^n \setminus O^*_{k+1}) - d(y, Q^j_{k+1}) \ge \sqrt{n}\ell(Q^j_{k+1}) - t > (2\sqrt{n}-1)t \ge t,$$

which implies that $(y,t) \in \widehat{O^*_{k+1}}$. Hence $G_1(x) = 0$, for all $x \in Q^j_{k+1}$.

On the other hand, take $x_j \in \mathbb{R}^n \setminus O^*_{k+1}$ such that $d(x_j, Q^j_{k+1}) \le 4\sqrt{n}\ell(Q^j_{k+1})$, and note that if $\ell(Q^j_{k+1})/2 \le t < \infty$ and $x \in Q^j_{k+1}$, then $B(x,t) \subset B(x_j, \alpha t)$, for $\alpha \ge 11\sqrt{n}$. Indeed, for $x_0 \in B(x,t)$, we have

$$|x_0 - x_j| \le |x_0 - x| + |x - x_j| < t + \sqrt{n}\ell(Q^j_{k+1}) + 4\sqrt{n}\ell(Q^j_{k+1}) \le t(1 + 2\sqrt{n} + 8\sqrt{n}) \le 11\sqrt{n}t.$$

Hence

$$G_2(x) \le \mathcal{A}^\alpha f_k(x_j) \le 2^{k+1}, \ \forall\, x \in Q^j_{k+1}.$$

Therefore, since $O_{k+1} \subset O^*_{k+1} \subset O^*_k$,

$$\begin{aligned}
II_k &\le 2^{k\left(\frac{p}{q}-1\right)} \int_{O^*_{k+1}} |\mathcal{A} f_k(x)||\psi(x)|w(x)dx \\
&= 2^{k\left(\frac{p}{q}-1\right)} \sum_{j\in\mathbb{N}} \int_{Q^j_{k+1}} |\mathcal{A} f_k(x)||\psi(x)|w(x)dx \\
&\le 2^{k\left(\frac{p}{q}-1\right)} \sum_{j\in\mathbb{N}} \left(\int_{Q^j_{k+1}} |G_1(x)||\psi(x)|w(x)dx + \int_{Q^j_{k+1}} |G_2(x)||\psi(x)|w(x)dx \right) \\
&\lesssim 2^{k\frac{p}{q}} \sum_{j\in\mathbb{N}} \int_{Q^j_{k+1}} |\psi(x)|w(x)dx = 2^{k\frac{p}{q}} \int_{O^*_{k+1}} |\psi(x)|w(x)dx \le 2^{k\frac{p}{q}} \int_{O^*_k} |\psi(x)|w(x)dx.
\end{aligned}$$

Now consider respectively $\mathcal{M}_d$ and $\mathcal{M}_c$ the dyadic maximal function and the centred maximal function over cubes. For some dimensional constant c_n, we have that $\mathcal{M}(\mathbf{1}_{O_k})(x) \le c_n \mathcal{M}_c(\mathbf{1}_{O_k})(x)$. Next, for each $k \in \mathbb{Z}$, we define the set $\widetilde{O}_{1-\gamma,k} := \left\{ x \in \mathbb{R}^n : \mathcal{M}_d(\mathbf{1}_{O_k})(x) > \frac{1-\gamma}{4^n c_n} \right\}$; and we take a Calderón-Zygmund decomposition of this set at height $\frac{1-\gamma}{4^n c_n}$: $\widetilde{O}_{1-\gamma,k} = \bigcup_{l\in\mathbb{N}} \widetilde{Q}^l_k$, where $\{\widetilde{Q}^l_k\}_{l\in\mathbb{N}}$ is a collection of disjoints dyadic cubes such that

$$\fint_{\widetilde{Q}^l_k} \mathbf{1}_{O_k}(x)dx \approx 1-\gamma.$$

Then, since

$$O^*_k \subset \left\{ x \in \mathbb{R}^n : \mathcal{M}_c(\mathbf{1}_{O_k})(x) > \frac{1-\gamma}{c_n} \right\} \subset \bigcup_{l\in\mathbb{N}} 2\widetilde{Q}^l_k,$$

(see [37, proof of Lemma 2.12]), for $r > r_w$, we have that

$$\begin{aligned}
\int_{O^*_k} |\psi(x)|w(x)dx &\le \sum_{l\in\mathbb{N}} \frac{1}{(1-\gamma)^r} \int_{2\widetilde{Q}^l_k} (1-\gamma)^r |\psi(x)|w(x)dx \\
&\approx \frac{1}{(1-\gamma)^r} \sum_{l\in\mathbb{N}} \int_{2\widetilde{Q}^l_k} \left(\fint_{\widetilde{Q}^l_k} \mathbf{1}_{O_k}(y)dy \right)^r |\psi(x)|w(x)dx \\
&\le \frac{1}{(1-\gamma)^r} \sum_{l\in\mathbb{N}} \int_{2\widetilde{Q}^l_k} \left(\fint_{\widetilde{Q}^l_k} \mathbf{1}_{O_k}(y)w(y)dy \right) \left(\fint_{\widetilde{Q}^l_k} w^{1-r'}(y)dy \right)^{r-1} |\psi(x)|w(x)dx \\
&\lesssim \frac{1}{(1-\gamma)^r} \sum_{l\in\mathbb{N}} \int_{2\widetilde{Q}^l_k} \left(\int_{\widetilde{Q}^l_k} \mathbf{1}_{O_k}(y)w(y)dy \right) w(\widetilde{Q}^l_k)^{-1} |\psi(x)|w(x)dx
\end{aligned}$$

$$\lesssim \frac{1}{(1-\gamma)^r} \sum_{l\in\mathbb{N}} \int_{\widetilde{Q}_k^l} \mathbf{1}_{O_k}(y)\mathcal{M}^w(\psi)(y)w(y)dy$$
$$\leq \frac{1}{(1-\gamma)^r} \int_{O_k} \mathcal{M}^w(\psi)(y)w(y)dy,$$

where

$$\mathcal{M}^w f(x) := \sup_{Q\ni x} \frac{1}{w(Q)} \int_Q |f(y)|w(y)dy. \tag{2.88}$$

Moreover, note that, since $1 < q' \leq \infty$, $\mathcal{M}^w : L^{q'}(w) \to L^{q'}(w)$ (recall that w can be seen as a doubling measure (1.11)). Therefore, by the estimates obtained for I and II, and by Proposition 2.43, we conclude, for q equal to p_0 or p_1,

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} 2^{k\left(\frac{p}{q}-1\right)}|f_k| \right\|_{T^q(w)} &\lesssim \sup_{\|\psi\|_{L^{q'}(w)}\leq 1} \sum_{k\in\mathbb{Z}} 2^{k\frac{p}{q}} \int_{O_k} \mathcal{M}^w(\psi)(x)w(x)dx \\
&\lesssim \sup_{\|\psi\|_{L^{q'}(w)}\leq 1} \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} \lambda^{\frac{p}{q}} \int_{O_k} \mathcal{M}^w(\psi)(x)w(x)dx\frac{d\lambda}{\lambda} \\
&\lesssim \sup_{\|\psi\|_{L^{q'}(w)}\leq 1} \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} \lambda^{\frac{p}{q}} \int_{\{x\in\mathbb{R}^n:\mathcal{A}^\alpha f(x)>\lambda\}} \mathcal{M}^w(\psi)(x)w(x)dx\frac{d\lambda}{\lambda} \\
&= \sup_{\|\psi\|_{L^{q'}(w)}\leq 1} \int_0^\infty \lambda^{\frac{p}{q}} \int_{\{x\in\mathbb{R}^n:\mathcal{A}^\alpha f(x)>\lambda\}} \mathcal{M}^w(\psi)(x)w(x)dx\frac{d\lambda}{\lambda} \\
&\approx \sup_{\|\psi\|_{L^{q'}(w)}\leq 1} \int_{\mathbb{R}^n} |\mathcal{A}^\alpha f(x)|^{\frac{p}{q}} \mathcal{M}^w(\psi)(x)w(x)dx \\
&\lesssim \sup_{\|\psi\|_{L^{q'}(w)}\leq 1} \|\mathcal{A}^\alpha f\|_{L^p(w)}^{\frac{p}{q}} \|\psi\|_{L^{q'}(w)} \lesssim \|\mathcal{A}f\|_{L^p(w)}^{\frac{p}{q}} = \|f\|_{T^p(w)}^{\frac{p}{q}} \leq 1,
\end{aligned}$$

which finishes the proof. □

Chapter 3
WEIGHTED BOUNDEDNESS OF OPERATORS

In Section 1.2 we defined conical square functions (1.26)-(1.31) and non-tangential maximal functions (1.32), associated with the elliptic operator in divergence form $L = -\mathrm{div}(A\nabla)$. The development of an appropriate Calderón-Zygmund theory for these operators (among others as functional calculus, Riesz transforms, etc.) has been of big interest after the resolution of the Kato conjecture in [6]. These operators cease to be classical Calderón-Zygmund operators, as their kernels do not have the required decay or smoothness. This causes, in particular, that their range of boundedness may no longer be the interval $(1, \infty)$ but some proper (small) bounded subinterval containing $p = 2$. P. Auscher, in a very nice monograph ([3]), obtained a new Calderón-Zygmund theory adapted to singular "non-integral" operators arising from elliptic operators (see [3] for historic remarks and references). A key ingredient in the method is the fact that in place of using kernels, which do not have reasonable behaviour, they use a representation of the operators in question in terms of the heat semigroup $\{e^{-tL}\}_{t>0}$ (or its gradient) that has some integral decay measured in terms of the so-called "off-diagonal" or Gaffney type estimates. The bottom line of [3] is that the operators under consideration are bounded precisely in the ranges where either the semigroup or its gradient has a nice behaviour.

After P. Auscher's fundamental monograph there has been quite a number of papers whose goal is to continue with the development of a generalized Calderón-Zygmund theory. We will mention some that are relevant for the present work. P.Auscher and J.M. Martell wrote a series of papers [9, 10, 11] where the weighted theory was developed and where some appropriate classes of Muckenhoupt weights were found. While vertical square functions (i.e., usual Littlewood-Paley-Stein functionals) behave as expected with and without weights (see, resp., [3, 11]), conical square functions have better ranges of boundedness in the unweighted case, even going beyond the intervals where the semigroup or its gradient has a nice behaviour, see [7].

In this chapter we contribute to the development of this generalized Calderón-Zygmund theory establishing boundedness and comparability of the square functions presented in (1.26)-(1.31) and of the non-tangential maximal functions (1.32) on weighted spaces $L^p(w)$ where $w \in A_\infty$. This was left open in [7] since some of the existing arguments naturally split the boundedness into the cases $p < 2$ and $p > 2$. That procedure, as learned from [11], is inefficient when adding weights, to obtain the right class of weights one has to be able to work with the whole interval where the unweighted estimates hold. Splitting the interval will lead to some distortion in the class of weights. To illustrate this, let us recall that in [7] it is shown that the square function $\mathrm{G_H}$, defined in (1.27) above, is bounded on $L^p(\mathbb{R}^n)$ for every $p_-(L) < p < \infty$ where $p_-(L)$, introduced in (1.21) above, is strictly smaller than 2. Using the approach in [7], and "stepping" at $p = 2$, this square function is bounded on $L^p(w)$ for every $2 < p < \infty$ and $w \in A_{p/2}$ (see the precise definitions above). However, as we will see in Theorem 3.1, one has boundedness on $L^p(w)$ for every $p_-(L) < p < \infty$ and $w \in A_{p/p_-(L)}$, hence in a bigger range and a wider class of weights since $p_-(L) < 2$. Moreover, the obtained class of weights is the natural one adapted to the unweighted range $(p_-(L), \infty)$, in view of the version of the Rubio de Francia extrapolation theorem in [9, Theorem 4.9] or Theorem 1.46, part (c). See also [24, 67] for related issues.

3.1 Conical square functions

Our first result establishes the boundedness of the square functions associated with the heat semigroup. Notice that when $w \equiv 1$, which corresponds to the unweighted case, this result recovers the estimates in the range $(p_-(L), \infty)$ obtained in [7].

Theorem 3.1. *Let $w \in A_\infty$.*

(*a*) $\mathcal{S}_{\mathrm{H}}$, G_{H}, *and* $\mathcal{G}_{\mathrm{H}}$ *are bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), \infty)$.*

(*b*) *Given $m \in \mathbb{N}$, $\mathcal{S}_{m,\mathrm{H}}$, $\mathrm{G}_{m,\mathrm{H}}$, and $\mathcal{G}_{m,\mathrm{H}}$ are bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), \infty)$.*

Equivalently, all the previous square functions are bounded on $L^p(w)$ for every $p_-(L) < p < \infty$ and every $w \in A_{\frac{p}{p_-(L)}}$.

The proof of Theorem 3.1 is split into two steps. First, we prove (*a*) (in doing that we only need to consider $\mathcal{G}_{\mathrm{H}}$ since $\mathrm{G}_{\mathrm{H}} f \leq \mathcal{G}_{\mathrm{H}} f$ and $\mathcal{S}_{\mathrm{H}} \leq 1/2\mathcal{G}_{\mathrm{H}}$). Second, we shall show that the square functions in (*b*) are all controlled by $\mathcal{S}_{\mathrm{H}}$ in $L^p(w)$ for every $w \in A_\infty$ and $0 < p < \infty$ (see Theorem 3.3). Gathering this and (*a*) will complete the proof of (*b*).

Our second result deals with the boundedness of the square functions related to the Poisson semigroup. Here the formulation is more involved since the ranges where these square functions are bounded not only depend on $p_-(L)$ and the weight but also on $p_+(L)$ and the parameter K. We also notice that when $w \equiv 1$ we recover the estimates obtained in [7].

Theorem 3.2. *Let $w \in A_\infty$.*

(*a*) *Given $K \in \mathbb{N}$, $\mathcal{S}_{K,\mathrm{P}}$ is bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), p_+(L)^{K,*})$.*

(*b*) *Given $K \in \mathbb{N}_0$, $\mathcal{G}_{K,\mathrm{P}}$ and $\mathrm{G}_{K,\mathrm{P}}$ are bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), p_+(L)^{K,*})$.*

The proof of this result is as follows. We will first show that each square function in (*a*) and (*b*) can be controlled by either $\mathcal{S}_{\mathrm{H}}$ or $\mathcal{G}_{\mathrm{H}}$ in $L^p(w)$ for every $w \in A_\infty$ and $p \in \mathcal{W}_w(0, p_+(L)^{K,*})$ (see Theorem 3.4). This, in concert with (*a*) in Theorem 3.1, will easily lead to the desired estimates.

We present the two aforementioned results containing the control of the previous square functions by $\mathcal{S}_{\mathrm{H}}$ and $\mathcal{G}_{\mathrm{H}}$. In the first result we deal with the square functions defined in terms of the heat semigroup, and in the second with the ones associated with the Poisson semigroup.

Theorem 3.3. *Given an arbitrary $f \in L^2(\mathbb{R}^n)$ there hold:*

(*a*) $\mathcal{S}_{\mathrm{H}}(x) \leq 1/2\mathcal{G}_{\mathrm{H}} f(x)$ *and* $\mathrm{G}_{m,\mathrm{H}} f(x) \leq \mathcal{G}_{m,\mathrm{H}} f(x)$, *for every $x \in \mathbb{R}^n$ and for all $m \in \mathbb{N}_0$.*

(*b*) *Given $m \in \mathbb{N}$, $\|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)}$, for all $w \in A_\infty$ and $0 < p < \infty$.*

(*c*) *Given $m \in \mathbb{N}$, $\|\mathcal{G}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)}$, for all $w \in A_\infty$ and $0 < p < \infty$.*

Theorem 3.4. *Given an arbitrary $f \in L^2(\mathbb{R}^n)$ there hold:*

(*a*) $\mathrm{G}_{K,\mathrm{P}} f(x) \leq \mathcal{G}_{K,\mathrm{P}} f(x)$, *for every $x \in \mathbb{R}^n$ and for all $K \in \mathbb{N}_0$.*

(*b*) *Given $K \in \mathbb{N}$, $\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)}$, for all $w \in A_\infty$ and $p \in \mathcal{W}_w(0, p_+(L)^{K,*})$.*

(*c*) $\|\mathcal{G}_{\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{G}_{\mathrm{H}} f\|_{L^p(w)}$, *for all $w \in A_\infty$ and $w \in \mathcal{W}_w(0, p_+(L)^*)$.*

(*d*) *Given $K \in \mathbb{N}$, $\|\mathcal{G}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)}$, for all $w \in A_\infty$ and $p \in \mathcal{W}_w(0, p_+(L)^{K,*})$.*

Let us observe that in (*b*) and (*d*) (and also (*c*) with $K = 0$), if $(2K + 1)\, p_+(L) \geq n$ the corresponding estimates hold for every $w \in A_\infty$ and every $0 < p < \infty$. Otherwise, if $(2K + 1)\, p_+(L) < n$, each corresponding estimate holds for all $0 < p < p_+(L)^{K,*}$ and $w \in RH_{(p_+(L)^{K,*}/p)'}$.

Proof of Theorem 3.1, part (a)**.**

Let us start by introducing more notation. From now on, Q_t denotes $t^2Le^{-t^2L}$, $t\nabla_y e^{-t^2L}$, or $t\nabla_{y,t}e^{-t^2L}$ in such a way that, if we write

$$\widetilde{\mathcal{A}}f(x) := \left(\iint_{\Gamma(x)} |Q_t f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

then $\widetilde{\mathcal{A}}f$ is respectively $\mathcal{S}_{\mathrm{H}}f$, $\mathrm{G}_{\mathrm{H}}f$, or $\mathcal{G}_{\mathrm{H}}f$.

The boundedness of $\widetilde{\mathcal{A}}$ follows from the combination of Proposition 2.69 and the following auxiliary result.

Proposition 3.5. *Let Q_t denote $t^2Le^{-t^2L}$, $t\nabla_y e^{-t^2L}$, or $t\nabla_{y,t}e^{-t^2L}$. If we set*

$$\widetilde{C}_{p_0}f(x) := \sup_{B\ni x}\left(\frac{1}{|B|}\int_B\left(\int_0^{r_B}\int_{B(x,t)} |Q_t f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p_0}{2}} dx\right)^{\frac{1}{p_0}},$$

then, for every $p_-(L) < p_0 \le 2$, there holds

$$\widetilde{C}_{p_0}f(x) \lesssim \mathcal{M}_{p_0}f(x), \qquad x\in\mathbb{R}^n. \tag{3.6}$$

Assuming this result momentarily we prove Theorem 3.1, part (a). Note that taking $F(y,t) = Q_t f(y)$ in (2.1) and in (2.68) we have that $\widetilde{\mathcal{A}}f(x) = \mathcal{A}F(x)$ and $\widetilde{C}_{p_0}f(x) = C_{p_0}F(x)$. Thus (3.6), in concert with (a) in Proposition 2.69, implies that, for every $0 < p < \infty$ and $w \in A_\infty$,

$$\|\widetilde{\mathcal{A}}f\|_{L^p(w)} \lesssim \|\widetilde{C}_{p_0}f\|_{L^p(w)} \lesssim \|\mathcal{M}_{p_0}f\|_{L^p(w)}, \quad \text{for all} \quad p_-(L) < p_0 \le 2,$$

provided $Q_t f \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. Hence, the above estimate holds for all functions $f \in L^\infty_c(\mathbb{R}^n)$. Next, fix $w \in A_\infty$ and $p \in \mathcal{W}_w(p_-(L),\infty)$. Then, there exists $p_-(L) < p_0 \le 2$ (close enough to $p_-(L)$), such that $w \in A_{\frac{p}{p_0}}$. Therefore, $\mathcal{M}_{p_0}$ is bounded on $L^p(w)$ and consequently the previous estimate leads to

$$\|\widetilde{\mathcal{A}}f\|_{L^p(w)} \le C\,\|f\|_{L^p(w)}, \qquad \forall\, f\in L^\infty_c(\mathbb{R}^n). \tag{3.7}$$

A routine density argument allows one to extend this estimate to all functions in $L^p(w)$.

Let us notice that (3.6) with $p_0 = 2$ appears implicit in [7, p. 5479]. Having used that estimate we would have obtain (3.7) for every $2 < p < \infty$ and $w \in A_{p/2}$. However, using C_{p_0} with p_0 very close to $p_-(L)$ allows to obtain better estimates: (3.7) holds for every $p_-(L) < p < \infty$ and $w \in A_{p/p_-(L)}$.

We are left with the proof of Proposition 3.5, in which we will use the following unweighted estimates for the conical square functions that we are currently considering.

Proposition 3.8. *The square functions $\mathcal{S}_{\mathrm{H}}$, G_{H}, and $\mathcal{G}_{\mathrm{H}}$ are bounded on $L^p(\mathbb{R}^n)$ for every $p_-(L) < p \le 2$.*

Let us note that the boundedness of G_{H} has been established in [3, Section 6.2]. On the other hand, one can easily see that $\mathcal{G}_{\mathrm{H}} \lesssim \mathrm{G}_{\mathrm{H}} + \mathcal{S}_{\mathrm{H}}$, and therefore we only have to consider $\mathcal{S}_{\mathrm{H}}$. In turn, this operator will be handled by using a Calderón-Zygmund type result from [11].

We would like to observe that, a posteriori, Theorem 3.1, part (a), applied with $w \equiv 1$, implies that $\mathcal{S}_{\mathrm{H}}$, G_{H}, and $\mathcal{G}_{\mathrm{H}}$ are also bounded on $L^p(\mathbb{R}^n)$ for every $2 \le p < \infty$ (and therefore in the range $p_-(L) \le p < \infty$). The case G_{H} was obtained in [7, Theorem 3.1, part (2)].

Proof of Proposition 3.5. Fix $p_-(L) < p_0 \le 2$ and $x_0 \in \mathbb{R}^n$. Take an arbitrary ball $B \ni x_0$ and split f into its local and global part: $f = f_{\mathrm{loc}} + f_{\mathrm{glob}} := f\mathbf{1}_{4B} + f\mathbf{1}_{\mathbb{R}^n\setminus 4B}$.

For f_{loc}, we use that $\widetilde{\mathcal{A}}$ is bounded on $L^{p_0}(\mathbb{R}^n)$ by Proposition 3.8:

$$\left(\frac{1}{|B|}\int_B\left(\int_0^{r_B}\int_{B(x,t)}|Q_t f_{\rm loc}(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p_0}{2}}dx\right)^{\frac{1}{p_0}}\leq\left(\frac{1}{|B|}\int_{\mathbb{R}^n}\widetilde{\mathcal{A}}f_{\rm loc}(x)^{p_0}\,dx\right)^{\frac{1}{p_0}}$$
$$\lesssim\left(\frac{1}{|B|}\int_{\mathbb{R}^n}|f_{\rm loc}(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}\lesssim\left(\frac{1}{|4B|}\int_{4B}|f(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}\lesssim\mathcal{M}_{p_0}f(x_0).$$

As for $f_{\rm glob}$, since $\{Q_t\}_{t>0}\in\mathcal{F}_\infty(L^{p_0}\to L^2)$ —where we recall that Q_t is $t^2Le^{-t^2L}$, $t\nabla_y e^{-t^2L}$, or $t\nabla_{y,t}e^{-t^2L}$— and since $\operatorname{supp} f_{\rm glob}\subset\mathbb{R}^n\setminus 4B$, we have

$$\left(\frac{1}{|B|}\int_B\left(\int_0^{r_B}\int_{B(x,t)}|Q_t f_{\rm glob}(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p_0}{2}}dx\right)^{\frac{1}{p_0}}$$
$$\lesssim\sum_{j\geq2}\left(\int_0^{r_B}\int_{2B}|Q_t(f\,\mathbf{1}_{C_j(B)})(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac12}$$
$$\lesssim\sum_{j\geq2}\left(\int_0^{r_B}\left(\int_{2^{j+1}B}|f(y)|^{p_0}\,dy\right)^{\frac{2}{p_0}}t^{-2n(\frac{1}{p_0}-\frac12)}e^{-c\frac{4^jr_B^2}{t^2}}\frac{dt}{t^{n+1}}\right)^{\frac12}$$
$$\lesssim\mathcal{M}_{p_0}f(x_0)\sum_{j\geq2}\left(\int_0^{r_B}(2^jr_B)^{\frac{2n}{p_0}}t^{-\frac{2n}{p_0}}e^{-c\frac{4^jr_B^2}{t^2}}\frac{dt}{t}\right)^{\frac12}\lesssim\mathcal{M}_{p_0}f(x_0).$$

Gathering the estimates obtained for $f_{\rm loc}$ and for $f_{\rm glob}$ gives us

$$\left(\frac{1}{|B|}\int_B\left(\int_0^{r_B}\int_{B(x,t)}|Q_t f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p_0}{2}}dx\right)^{\frac{1}{p_0}}\lesssim\mathcal{M}_{p_0}f(x_0).$$

Taking the supremum over all balls B such that $x_0\in B$ we readily conclude the desired estimate. □

Proof of Proposition 3.8. As explained above we only need to consider the operator $\mathcal{S}_{\rm H}$. It is well-known that $\mathcal{S}_{\rm H}$ is bounded on $L^2(\mathbb{R}^n)$. Fix then $p_-(L)<p<2$ and take $p_-(L)<p_0<p<2$. We shall apply [11, Theorem 2.4] (see also [3] and [9]). We claim that given $f\in L^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp} f\subset B\subset\mathbb{R}^n$, the following estimates hold

$$\left(\fint_{C_j(B)}\left|\mathcal{S}_{\rm H}(I-e^{-r_B^2L})^Mf\right|^{p_0}dx\right)^{\frac{1}{p_0}}\leq g(j)\left(\fint_B|f|^{p_0}\,dx\right)^{\frac{1}{p_0}},\qquad j\geq2,\tag{3.9}$$

and

$$\left(\fint_{C_j(B)}\left|I-(I-e^{-r_B^2L})^Mf\right|^{2}dx\right)^{\frac{1}{2}}\leq g(j)\left(\fint_B|f|^{p_0}\,dx\right)^{\frac{1}{p_0}},\qquad j\geq1,\tag{3.10}$$

with $g(j)=C\,2^{-j(2M+\frac{n}{p_0})}$. Assuming this and taking M large enough in such a way that $\sum_{j\geq1}g(j)2^{jn}<\infty$, [11, Theorem 2.4] implies that $\mathcal{S}_{\rm H}$ is of weak-type (p_0,p_0) and, by Marcinkiewicz's interpolation Theorem, bounded on $L^p(\mathbb{R}^n)$, which is our goal.

In view of the previous considerations we need to obtain (3.9) and (3.10). Fix a ball B. For $f\in L^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp} f\subset B$, we first prove (3.9). Define $A_{r_B^2}:=(I-e^{-r_B^2L})^M$ and by Fubini (or see [32, Lemma 1]) conclude that

$$\left(\fint_{C_j(B)}\left|\mathcal{S}_{\rm H}A_{r_B^2}f(x)\right|^{p_0}dx\right)^{\frac{1}{p_0}}\leq\left(\fint_{C_j(B)}\left|\mathcal{S}_{\rm H}A_{r_B^2}f(x)\right|^{2}dx\right)^{\frac12}$$

$$
\begin{aligned}
&\lesssim |2^jB|^{-\frac{1}{2}} \left(\iint_{\mathcal{R}(\overline{C_j(B)})} \left| t^2 L e^{-t^2 L} A_{r_B^2} f(y) \right|^2 \frac{dy\, dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim |2^jB|^{-\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus 2^{j-1}B} \int_0^\infty \left| t^2 L e^{-t^2 L} A_{r_B^2} f(y) \right|^2 \frac{dt\, dy}{t} \right)^{\frac{1}{2}} \\
&\qquad + |2^jB|^{-\frac{1}{2}} \left(\int_{2^{j-1}\,B} \int_{2^{j-1}r_B}^\infty \left| t^2 L e^{-t^2 L} A_{r_B^2} f(y) \right|^2 \frac{dt\, dy}{t} \right)^{\frac{1}{2}} \\
&=: |2^jB|^{-\frac{1}{2}} (I + II).
\end{aligned}
$$

We estimate each term in turn. Before that, let us recall the following off-diagonal estimate obtained in [54, p. 504]:

$$
\left\| \frac{s^2}{t^2} \big(e^{-s^2 L} - e^{-(s^2+t^2)L} \big) (f\, \mathbf{1}_E) \right\|_{L^2(F)} \le C\, e^{-c\, \frac{d(E,F)^2}{s^2}} \|f\|_{L^2(E)}, \qquad 0 < t \le s, \tag{3.11}
$$

with C independent of t and s. This and Lemma 1.35 imply that for every $M \ge 1$ there exists C such that for every $0 < t \le s$ there holds

$$
\left\| s^2 L\, e^{-s^2 L} \left(\frac{s^2}{t^2} \right)^M \big(e^{-s^2 L} - e^{-(s^2+t^2)L} \big)^M (f\, \mathbf{1}_E) \right\|_{L^2(F)} \le C\, s^{-n\left(\frac{1}{p_0} - \frac{1}{2}\right)} e^{-c\, \frac{d(E,F)^2}{s^2}} \|f\|_{L^{p_0}(E)}. \tag{3.12}
$$

After these preparations we estimate II. Doing the change of variables $t = \sqrt{M+1}\, s$ and using (3.12), easy calculations lead us to obtain

$$
\begin{aligned}
II &\lesssim \left(\int_{c2^j r_B}^\infty \left\| s^2 L e^{-s^2 L} \big(e^{-s^2 L} - e^{-(s^2 + r_B^2)L} \big)^M f \right\|_{L^2(2^{j-1}\,B)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&\qquad \lesssim \left(\int_{c2^j r_B}^\infty \left(\frac{r_B}{s} \right)^{4M} s^{-2n\left(\frac{1}{p_0} - \frac{1}{2}\right)} \frac{ds}{s} \right)^{\frac{1}{2}} \|f\|_{L^{p_0}(B)} \lesssim 2^{-j\left(2M + \frac{n}{p_0}\right)} |2^jB|^{\frac{1}{2}} \left(\fint_B |f(x)|^{p_0}\, dx \right)^{\frac{1}{p_0}}.
\end{aligned}
$$

Let us next estimate I. We proceed as in [54] or [55, p. 53-56]. Changing variables as before we obtain

$$
\begin{aligned}
I &\lesssim \left(\int_0^{r_B} \left\| s^2 L e^{-(M+1)s^2 L} \big(I - e^{-r_B^2 L} \big)^M f \right\|_{L^2(\mathbb{R}^n \setminus 2^{j-1}B)}^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\qquad + \left(\int_{r_B}^\infty \left\| s^2 L e^{-s^2 L} \big(e^{-s^2 L} - e^{-(s^2 + r_B^2)L} \big)^M f \right\|_{L^2(\mathbb{R}^n \setminus 2^{j-1}B)}^2 \frac{dt}{t} \right)^{\frac{1}{2}} =: I_1 + I_2.
\end{aligned}
$$

For I_2, we employ (3.12) and conclude that

$$
I_2 \lesssim \left(\int_{r_B}^\infty \left(\frac{r_B}{s} \right)^{4M} s^{-2n\left(\frac{1}{p_0} - \frac{1}{2}\right)} e^{-c\, \frac{4^j r_B^2}{s^2}} \frac{ds}{s} \right)^{\frac{1}{2}} \|f\|_{L^{p_0}(B)} \lesssim 2^{-j\left(2M + \frac{n}{p_0}\right)} |2^jB|^{\frac{1}{2}} \left(\fint_B |f(x)|^{p_0}\, dx \right)^{\frac{1}{p_0}}.
$$

For I_1, expanding $\big(I - e^{-r_B^2 L} \big)^M$ and using the $L^{p_0}(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimates satisfied by the heat semigroup, we get

$$
\begin{aligned}
I_1 &\lesssim \left(\int_0^{r_B} \left\| s^2 L e^{-(M+1)s^2 L} f \right\|_{L^2(\mathbb{R}^n \setminus 2^{j-1}B)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} + \sup_{1 \le k \le M} \left(\int_0^{r_B} \left\| s^2 L e^{-\left((M+1)s^2 + k r_B^2\right)L} f \right\|_{L^2(\mathbb{R}^n \setminus 2^{j-1}B)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^{r_B} s^{-2n\left(\frac{1}{p_0} - \frac{1}{2}\right)} e^{-c\, \frac{4^j r_B^2}{s^2}} \frac{ds}{s} \right)^{\frac{1}{2}} \|f\|_{L^{p_0}(B)}
\end{aligned}
$$

$$+ \sup_{1\le k\le M}\left(\int_0^{r_B}\left(\frac{s^2}{(M+1)s^2+kr_B^2}\right)^2\left((M+1)s^2+kr_B^2\right)^{-n\left(\frac{1}{p_0}-\frac{1}{2}\right)}e^{-c\frac{4^j r_B^2}{(M+1)s^2+kr_B^2}}\frac{ds}{s}\right)^{\frac{1}{2}}\|f\|_{L^{p_0}(B)}$$

$$\lesssim e^{-c\,4^j}|2^jB|^{\frac{1}{2}}\left(\fint_B|f(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}+e^{-c\,4^j}|2^jB|^{\frac{1}{2}}\left(\fint_B|f(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}\left(\int_0^{r_B}\left(\frac{s^2}{r_B^2}\right)^2\frac{ds}{s}\right)^{\frac{1}{2}}$$

$$\lesssim e^{-c\,4^j}|2^jB|^{\frac{1}{2}}\left(\fint_B|f(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}.$$

Gathering all the estimates that we have obtained allows us to complete the proof of (3.9):

$$\left(\fint_{C_j(B)}\left|\mathcal{S}_{\mathrm{H}}A_{r_B^2}f(x)\right|^{p_0}dx\right)^{\frac{1}{p_0}}\le C2^{-j(2M+\frac{n}{p_0})}\left(\fint_B|f(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}.$$

To prove (3.10), we use that $\{e^{-t^2L}\}_{t>0}\in\mathcal{F}_\infty(L^{p_0}\to L^2)$ and for every $j\ge 1$

$$\left(\fint_{C_j(B)}|I-(I-e^{-r_B^2L})^Mf|^2\right)^{\frac{1}{2}}\le\sum_{k=1}^M C_{k,M}\left(\fint_{C_j(B)}|e^{-kr_B^2L}f|^2\right)^{\frac{1}{2}}\lesssim e^{-c4^j}\left(\fint_B|f(x)|^{p_0}\,dx\right)^{\frac{1}{p_0}}.$$

□

Proof of Theorem 3.1, part (b)**.**

Take $w\in A_\infty$, $m\in\mathbb{N}$, and $f\in L_c^\infty(\mathbb{R}^n)$, and apply Theorem 3.3. Then, for all $0<p<\infty$,

$$\|\mathrm{G}_{m,\mathrm{H}}f\|_{L^p(w)}\lesssim\|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)},\quad\|\mathcal{G}_{m,\mathrm{H}}f\|_{L^p(w)}\lesssim\|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)},\quad\text{and}\quad\|\mathcal{S}_{m,\mathrm{H}}f\|_{L^p(w)}\lesssim\|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)}.$$

Now, use Theorem 3.1 part (a) to conclude that for all $p\in\mathcal{W}_w(p_-(L),\infty)$

$$\|\mathrm{G}_{m,\mathrm{H}}f\|_{L^p(w)}\lesssim\|f\|_{L^p(w)},\quad\|\mathcal{G}_{m,\mathrm{H}}f\|_{L^p(w)}\lesssim\|f\|_{L^p(w)},\quad\text{and}\quad\|\mathcal{S}_{m,\mathrm{H}}f\|_{L^p(w)}\lesssim\|f\|_{L^p(w)},$$

for all $f\in L_c^\infty(\mathbb{R}^n)$. By a standard density argument these estimates easily extend to all functions $f\in L^p(w)$. □

Proof of Theorem 3.2, part (a)**.**

From Theorem 3.4 part (b), given $w\in A_\infty$, we have for all $K\in\mathbb{N}$, $p\in\mathcal{W}_w(0,p_+(L)^{K,*})$, and $f\in L_c^\infty(\mathbb{R}^n)$

$$\|\mathcal{S}_{K,\mathrm{P}}f\|_{L^p(w)}\lesssim\|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)}.$$

Hence, applying Theorem 3.1 part (a), we obtain, for all $p\in\mathcal{W}_w(p_-(L),p_+(L)^{K,*})$,

$$\|\mathcal{S}_{K,\mathrm{P}}f\|_{L^p(w)}\lesssim\|f\|_{L^p(w)},\qquad f\in L_c^\infty(\mathbb{R}^n).$$

A density argument allows us to complete the proof. □

Proof of Theorem 3.2, part (b)**.**

Take $w \in A_\infty$ and apply Theorem 3.4 parts (a), (c), and (d) to obtain, for all $K \in \mathbb{N}$, $p \in \mathcal{W}_w(0, p_+(L)^{K,*})$, and $f \in L_c^\infty(\mathbb{R}^n)$

$$\|\mathrm{G}_{K,\mathrm{P}}f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)} \quad \text{and} \quad \|\mathcal{G}_{K,\mathrm{P}}f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)},$$

and

$$\|\mathrm{G}_{\mathrm{P}}f\|_{L^p(w)} \lesssim \|\mathcal{G}_{\mathrm{H}}f\|_{L^p(w)} \quad \text{and} \quad \|\mathcal{G}_{\mathrm{P}}f\|_{L^p(w)} \lesssim \|\mathcal{G}_{\mathrm{H}}f\|_{L^p(w)}.$$

Now, apply Theorem 3.1, part (a), and conclude, by a density argument, that for all $p \in \mathcal{W}_w(p_-(L), p_+(L)^{K,*})$, $K \in \mathbb{N}_0$, and $f \in L^p(w)$, there hold

$$\|\mathrm{G}_{K,\mathrm{P}}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \quad \text{and} \quad \|\mathcal{G}_{K,\mathrm{P}}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

□

Proof of Theorem 3.3, part (a)**.**

It follows immediately from the following facts:

$$\left|t\nabla_y (t^2L)^m e^{-t^2L} f(y)\right| \le \left|t\nabla_{y,t} (t^2L)^m e^{-t^2L} f(y)\right|, \text{ for all } t>0,\ y \in \mathbb{R}^n, \text{ and } m \in \mathbb{N}_0;$$

and

$$\left|t^2 L e^{-t^2L} f(y)\right| \le \frac{\left|t\nabla_{y,t} e^{-t^2L} f(y)\right|}{2}, \text{ for all } t>0, \text{ and } y \in \mathbb{R}^n.$$

□

Proof of Theorem 3.3, part (b)**.**

For $m = 1$ there is nothing to prove. So, take $m \in \mathbb{N}$ such that $m \ge 2$ and consider

$$T_{\frac{t^2}{2}} := \left(\tfrac{t^2}{2}L\right)^{m-1} e^{-\frac{t^2}{2}L}.$$

Fix $0 < p < \infty$ and $w \in A_\infty$. Pick $r \ge \max\{\frac{p}{2}, r_w\}$ so that $w \in A_r$ and $0 < p \le 2r$. Then, since $\{(t^2L)^m e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^2)$ and applying Proposition 2.43 in the next-to-last inequality, we have

$$\begin{aligned}
\|\mathcal{S}_{m,\mathrm{H}}f\|_{L^p(w)} &\lesssim \left(\int_{\mathbb{R}^n} \left(\iint_{\Gamma(x)} \left| T_{\frac{t^2}{2}} \left(\tfrac{t^2}{2} L e^{-\frac{t^2}{2}L} f\right)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\ge1} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} \left| T_{\frac{t^2}{2}} \left(\left(\tfrac{t^2}{2} L e^{-\frac{t^2}{2}L} f\right)\mathbf{1}_{C_j(B(x,t))}\right)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\ge1} e^{-c4^j} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,2^{j+1}t)} \left| \tfrac{t^2}{2} L e^{-\frac{t^2}{2}L} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\ge1} e^{-c4^j} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,2^{j+1}\sqrt{2}t)} \left| t^2 L e^{-t^2L} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\ge1} 2^{j\frac{nr}{p}} e^{-c4^j} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} \left| t^2 L e^{-t^2L} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)}. \quad \square
\end{aligned}$$

Proof of Theorem 3.3, part (c)**.**

Take $m \in \mathbb{N}$ and consider

$$A_{\frac{t^2}{2}} := \frac{t}{\sqrt{2}} \nabla_{y,t} e^{-\frac{t^2}{2}L} \quad \text{and} \quad B_{\frac{t^2}{2},m} := \left(\frac{t^2}{2}L\right)^m e^{-\frac{t^2}{2}L}.$$

Fix $0 < p < \infty$ and $w \in A_\infty$. Pick $r \geq \max\{\frac{p}{2}, r_w\}$ so that $w \in A_r$ and $0 < p \leq 2r$. Then, applying the $L^2(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimates satisfied by $\{t\nabla_{y,t} e^{-t^2 L}\}_{t>0}$ and Proposition 2.43, we obtain

$$\begin{aligned}
\|\mathcal{G}_{m,\mathrm{H}} f\|_{L^p(w)} &\lesssim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} \left|A_{\frac{t^2}{2}} B_{\frac{t^2}{2},m} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\geq 1} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} \left|A_{\frac{t^2}{2}} \left(\left(B_{\frac{t^2}{2},m} f\right) \mathbf{1}_{C_j(B(x,t))}\right)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,2^{j+1}t)} \left|B_{\frac{t^2}{2},m} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,2^{j+1}\sqrt{2}t)} \left|B_{t^2,m} f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)\,dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} 2^{j\frac{nr}{p}} \|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)} \\
&\lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)},
\end{aligned}$$

where in the last inequality we have used part (b). □

Proof of Theorem 3.4, part (a)**.**

It is enough to observe the following:

$$\left|t\nabla_y (t\sqrt{L})^{2K} e^{-t\sqrt{L}} f(y)\right| \leq \left|t\nabla_{y,t} (t\sqrt{L})^{2K} e^{-t\sqrt{L}} f(y)\right|, \text{ for all } t > 0,\ y \in \mathbb{R}^n, \text{ and } K \in \mathbb{N}_0.$$

□

Proof of Theorem 3.4, part (b)**.**

In view of Theorem 3.3, part (b), it is enough to show that

$$\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^p(w)}. \tag{3.13}$$

for all $w \in A_\infty$, $K \in \mathbb{N}$, and $p \in \mathcal{W}_w(0, p_+(L)^{K,*})$. Furthermore, by Theorem 1.46, part (b), (or part (e) if $p_+(L)^{K,*} = \infty$), it suffices to prove the above inequality for some fix p in $(0, p_+(L)^{K,*})$ and $w \in RH_{\left(\frac{p_+(L)^{K,*}}{p}\right)'}$. In particular we can take $p = 2$. Hence, we need to show that

$$\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^2(w)} \lesssim \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^2(w)}. \tag{3.14}$$

for all $K \in \mathbb{N}$ and $w \in RH_{\left(\frac{p_+(L)^{K,*}}{2}\right)'}$. To this end, we set

$$B_{t,K} := \left(t^2 L\right)^K e^{-t^2 L},$$

and apply the subordination formula (1.36) and Minkowski's inequality:

$$
\begin{aligned}
\left\|\mathcal{S}_{K,\mathrm{P}}f\right\|_{L^2(w)} &\lesssim \left(\int_{\mathbb{R}^n}\iint_{\Gamma(x)}\left|(t^2L)^K\int_0^\infty e^{-u}u^{\frac12}e^{-\frac{t^2}{4u}L}f(y)\,\frac{du}{u}\right|^2\frac{dy\,dt}{t^{n+1}}\,w(x)\,dx\right)^{\frac12}\\
&\lesssim \int_0^\infty e^{-u}u^{\frac12}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)}\left|(t^2L)^K e^{-\frac{t^2}{4u}L}f(y)\right|^2\frac{dy\,dt}{t^{n+1}}w(x)\,dx\right)^{\frac12}\frac{du}{u}\\
&\lesssim \int_0^{\frac14} e^{-u}u^{\frac12}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)}\left|(t^2L)^K e^{-\frac{t^2}{4u}L}f(y)\right|^2\frac{dy\,dt}{t^{n+1}}w(x)\,dx\right)^{\frac12}\frac{du}{u}\\
&\quad+\int_{\frac14}^\infty e^{-u}u^{\frac12}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)}\left|(t^2L)^K e^{-\frac{t^2}{4u}L}f(y)\right|^2\frac{dy\,dt}{t^{n+1}}w(x)\,dx\right)^{\frac12}\frac{du}{u}\\
&=: I+II.
\end{aligned}
$$

In order to estimate I, we distinguish two cases. Assume first that $n\le(2K+1)p_+(L)$. Under this assumption, we take $s>s_w$ and $\max\left\{2,\frac{2sp_+(L)}{p_+(L)+2s}\right\}<\widetilde{q}<\min\{p_+(L),2s\}$, (if $p_+(L)=\infty$ take $\widetilde{q}:=2s$). Then, we have that $2<\widetilde{q}<p_+(L)$, $\frac{\widetilde{q}}{2}\le s<\infty$, and $w\in RH_{s'}$. Besides, note that from our choices of s and $\widetilde{q}$, we have that

$$
K+\frac12+\frac{n}{4s}-\frac{n}{2\widetilde{q}}>K+\frac12-\frac{n}{2p_+(L)}\ge 0.
$$

Consider now the case $n>(2K+1)p_+(L)$. Then, the condition $w\in RH_{\left(\frac{p_+(L)_{K,*}}{2}\right)'}$ implies $s_w<\frac{p_+(L)n}{2(n-(2K+1)p_+(L))}$. Therefore, it is possible to pick $\varepsilon_1>0$ small enough and $2<\widetilde{q}<p_+(L)$ so that

$$
s_w<\frac{\widetilde{q}n}{2(1+\varepsilon_1)(n-(2K+1)\widetilde{q})}.
$$

Besides, since $\widetilde{q}<\widetilde{q}n/(n-(2K+1)\widetilde{q})$ there also exists $\varepsilon_2>0$ so that

$$
\widetilde{q}<\frac{\widetilde{q}n}{(1+\varepsilon_2)(n-(2K+1)\widetilde{q})}.
$$

Take $\varepsilon_0:=\min\{\varepsilon_1,\varepsilon_2\}$ and $s:=\frac{\widetilde{q}n}{2(1+\varepsilon_0)(n-(2K+1)\widetilde{q})}$. Then our choices guarantee that $2<\widetilde{q}<p_+(L)$, $\frac{\widetilde{q}}{2}\le s<\infty$, and $w\in RH_{s'}$; and that

$$
K+\frac12+\frac{n}{4s}-\frac{n}{2\widetilde{q}}=\varepsilon_0\left(\frac{n}{2\widetilde{q}}-K-\frac12\right)>\varepsilon_0\left(\frac{n}{2p_+(L)}-K-\frac12\right)>0.
$$

Then applying Jensen's inequality to the integral in y, we have

$$
\begin{aligned}
I &\lesssim \int_0^{\frac14} e^{-u}u^{K+\frac12}\left(\int_{\mathbb{R}^n}\int_0^\infty\left(\int_{B(x,t)}\left|B_{\frac{t}{2\sqrt{u}},K}f(y)\right|^{\widetilde{q}}dy\right)^{\frac{2}{\widetilde{q}}}\frac{dt}{t^{\frac{2n}{\widetilde{q}}+1}}w(x)\,dx\right)^{\frac12}\frac{du}{u}\\
&=: \int_0^{\frac14} e^{-u}u^{K+\frac12}\left(\int_{\mathbb{R}^n}|\mathcal{J}(u,x)|^2\,w(x)\,dx\right)^{\frac12}\frac{du}{u}. \qquad (3.15)
\end{aligned}
$$

Fix $0<u<\frac14$. Note that since $1<\frac{\widetilde{q}}{2}\le s<\infty$, for $\alpha:=2\sqrt{u}\in(0,1]$ and $q:=\frac{\widetilde{q}}{2}$ we can apply Proposition 2.61 and conclude,

$$
\int_{\mathbb{R}^n}|\mathcal{J}(u,x)|^2\,w(x)\,dx=\int_0^\infty\int_{\mathbb{R}^n}\left(\int_{B(x,2\sqrt{u}\frac{t}{2\sqrt{u}})}\left|B_{\frac{t}{2\sqrt{u}},K}f(y)\right|^{\widetilde{q}}dy\right)^{\frac{2}{\widetilde{q}}}w(x)dx\frac{dt}{t^{\frac{2n}{\widetilde{q}}+1}}
$$

$$
\begin{aligned}
&\lesssim u^{\frac{n}{2s}} \int_{\mathbb{R}^n} \int_0^\infty \left(\int_{B(x,\frac{t}{2\sqrt{u}})} \left| B_{\frac{t}{2\sqrt{u}},K} f(y) \right|^{\widetilde{q}} dy \right)^{\frac{2}{\widetilde{q}}} \frac{dt}{t^{\frac{2n}{\widetilde{q}}+1}}\, w(x)dx \\
&\lesssim u^{\frac{n}{2s}-\frac{n}{\widetilde{q}}} \int_{\mathbb{R}^n} \int_0^\infty \left(\int_{B(x,t)} \left| B_{t,K} f(x) \right|^{\widetilde{q}} dy \right)^{\frac{2}{\widetilde{q}}} \frac{dt}{t^{\frac{2n}{\widetilde{q}}+1}} w(x)dx \\
&=: u^{\frac{n}{2s}-\frac{n}{\widetilde{q}}} \int_{\mathbb{R}^n} \mathcal{T}(x)^2\, w(x)dx, \qquad (3.16)
\end{aligned}
$$

where in the last inequality we have changed the variable t into $2\sqrt{u}t$. Applying now $L^2(\mathbb{R}^n) - L^{\widetilde{q}}(\mathbb{R}^n)$ off-diagonal estimates and Proposition 2.43, we can bound the last integral above as follows

$$
\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{T}(x)^2\, w(x)dx &\lesssim \int_{\mathbb{R}^n} \int_0^\infty \left(\int_{B(x,t)} \left| e^{-\frac{t^2}{2}L} B_{\frac{t}{\sqrt{2}},K} f(x) \right|^{\widetilde{q}} dy \right)^{\frac{2}{\widetilde{q}}} \frac{dt}{t^{\frac{2n}{\widetilde{q}}+1}} w(x)dx \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2^{j+1}t)} \left| B_{\frac{t}{\sqrt{2}},K} f(x) \right|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2^{j+1}\sqrt{2}t)} \left| B_{t,K} f(x) \right|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \\
&\lesssim \sum_{j\geq 1} 2^{jnr} e^{-c4^j} \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^2(w)}^2 \\
&\lesssim \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^2(w)}^2,
\end{aligned}
$$

where $r > r_w$. This and (3.16) yield

$$
I \lesssim \int_0^{\frac{1}{4}} e^{-u} u^{K+\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\mathcal{J}(u,x)|^2 w(x)dx \right)^{\frac{1}{2}} \frac{du}{u} = \int_0^{\frac{1}{4}} e^{-u} u^{K+\frac{1}{2}+\frac{n}{4s}-\frac{n}{2\widetilde{q}}} \frac{du}{u} \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^2(w)} \lesssim \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^2(w)}.
$$

We finally estimate II. Pick $1 \leq r < \infty$ such that $w \in A_r$. Hence, since $1 < 2\sqrt{u}$, changing the variable t into $2\sqrt{u}t$, applying Minkowski's integral inequality, and Proposition 2.43 for $\alpha = 1$ and $\beta = 2\sqrt{u}$, we obtain

$$
\begin{aligned}
II &\lesssim \int_{\frac{1}{4}}^\infty e^{-u} u^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,t)} |(t^2L)^K e^{-\frac{t^2}{4u}L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \right)^{\frac{1}{2}} \frac{du}{u} \\
&\approx \int_{\frac{1}{4}}^\infty e^{-u} u^{K+\frac{1}{2}-\frac{n}{4}} \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2\sqrt{u}t)} |(t^2L)^K e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \right)^{\frac{1}{2}} \frac{du}{u} \\
&\lesssim \int_{\frac{1}{4}}^\infty e^{-u} u^{K+\frac{1}{2}-\frac{n}{4}} \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2\sqrt{u}t)} |(t^2L)^K e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \right)^{\frac{1}{2}} \frac{du}{u} \\
&\lesssim \int_{\frac{1}{4}}^\infty e^{-u} u^{K+\frac{1}{2}-\frac{n}{4}+\frac{nr}{4}} \frac{du}{u} \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^2(w_0)} \lesssim \|\mathcal{S}_{K,\mathrm{H}} f\|_{L^2(w)}.
\end{aligned}
$$

This estimate, together with the one obtained for I and the comments done above, allows us to finish the proof. □

Proof of Theorem 3.4, parts (c) and (d).

We first invoke [7, Lemma 3.5]: for every $K \in \mathbb{N}_0$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, there holds

$$
\mathcal{G}_{K,\mathrm{P}} f(x) \lesssim K \left(\int_0^\infty \int_{B(x,2t)} |(t^2L)^K e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}
$$

$$+\left(\int_0^\infty\int_{B(x,2t)}|t\nabla_{y,t}(t^2L)^Ke^{-t^2L}f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}$$

$$+\left(\int_0^\infty\int_{B(x,2t)}|(t^2L)^K(e^{-t\sqrt{L}}-e^{-t^2L})f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}. \tag{3.17}$$

The first and the second term in the right-hand side of the above inequality will be easily controlled in $L^p(w)$, applying Proposition 2.43, by $\mathcal{S}_{K,\mathrm{H}}f$ and $\mathcal{G}_{K,\mathrm{H}}f$ respectively. So, we just need to deal with the third term. To this end we define

$$\mathfrak{G}_{K,\mathrm{P}}f(x):=\left(\int_0^\infty\int_{B(x,2t)}|(t^2L)^K(e^{-t\sqrt{L}}-e^{-t^2L})f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

We claim that for all $K\in\mathbb{N}_0$, $w\in A_\infty$, and $p\in\mathcal{W}_w(0,p_+(L)^{K,*})$, the following estimate holds:

$$\|\mathfrak{G}_{K,\mathrm{P}}f\|_{L^p(w)}\lesssim\|\mathcal{S}_{K+1,\mathrm{H}}f\|_{L^p(w)}. \tag{3.18}$$

Assuming this momentarily and applying Proposition 2.43 to the first two terms in the right-hand side of (3.17) we conclude, for all $K\in\mathbb{N}_0$, $w\in A_\infty$, and $p\in\mathcal{W}_w(0,p_+(L)^{K,*})$,

$$\|\mathcal{G}_{K,\mathrm{P}}f\|_{L^p(w)}\lesssim K\|\mathcal{S}_{K,\mathrm{H}}f\|_{L^p(w)}+\|\mathcal{G}_{K,\mathrm{H}}f\|_{L^p(w)}+\|\mathcal{S}_{K+1,\mathrm{H}}f\|_{L^p(w)}. \tag{3.19}$$

For $K\in\mathbb{N}$, apply Theorem3.3 parts (b) and (c). This proves part (d). To obtain part (c), we take $K=0$ in (3.19). Note that clearly $\mathcal{S}_{\mathrm{H}}f\leq 1/2\mathcal{G}_{\mathrm{H}}f$ and therefore

$$\|\mathcal{G}_{\mathrm{P}}f\|_{L^p(w)}\lesssim\|\mathcal{G}_{\mathrm{H}}f\|_{L^p(w)}+\|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)}\lesssim\|\mathcal{G}_{\mathrm{H}}f\|_{L^p(w)}.$$

To complete the proof we need to obtain (3.18). But note that, as in the proof of Theorem 3.4, part (b), by Theorem 1.46, part (b), (or part (e) if $p_+(L)^{K,*}=\infty$), it suffices to show (3.18) for $p=2$.

Given $K\in\mathbb{N}_0$ and $w\in RH_{\left(\frac{p_+(L)^{K,*}}{2}\right)'}$, we apply the subordination formula (1.36) and Minkowski's inequality:

$$\begin{aligned}
\|\mathfrak{G}_{K,\mathrm{P}}f\|_{L^2(w)}&\lesssim\int_0^\infty e^{-u}u^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2t)}|(t^2L)^K(e^{-\frac{t^2}{4u}L}-e^{-t^2L})f(y)|^2\frac{dy\,dt}{t^{n+1}}w(x)dx\right)^{\frac{1}{2}}\frac{du}{u}\\
&=:\int_0^\infty e^{-u}u^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|F(x,u)|^2w(x)dx\right)^{\frac{1}{2}}\frac{du}{u}\\
&\lesssim\int_0^{\frac{1}{4}} e^{-u}u^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|F(x,u)|^2w(x)dx\right)^{\frac{1}{2}}\frac{du}{u}+\int_{\frac{1}{4}}^\infty e^{-u}u^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|F(x,u)|^2w(x)dx\right)^{\frac{1}{2}}\frac{du}{u}\\
&=:I+II.
\end{aligned}$$

Again, as in the proof of Theorem 3.4, part (b), we can find $\widetilde{q}$ and s such that $2<\widetilde{q}<p_+(L)$, $\frac{\widetilde{q}}{2}\leq s<\infty$, $w\in RH_{s'}$, and

$$\theta:=K+\frac{1}{2}+\frac{n}{4s}-\frac{n}{2\widetilde{q}}>0.$$

For later use, we choose $0<\widetilde{\theta}<\min\{4\theta,1\}$ so that for every $0<a<1$,

$$\int_a^1 t^{4\theta-1}\frac{dt}{t}\leq\int_a^1 t^{\widetilde{\theta}-1}\frac{dt}{t}\lesssim a^{\widetilde{\theta}-1}. \tag{3.20}$$

Fix now $0 < u < \frac{1}{4}$, and note that

$$\big|(e^{-\frac{t^2}{4u}L} - e^{-t^2L})f\big| \lesssim \int_t^{\frac{t}{2\sqrt{u}}} \big|r^2Le^{-r^2L}f\big|\frac{dr}{r}.$$

We set $H_K(y,r) := (r^2L)^{K+1}e^{-r^2L}f(y)$. Using the previous estimate and applying Minkowski's and Jensen's inequalities, it follows that

$$\begin{aligned}
F(x,u) &\lesssim \left(\int_0^\infty \left(\int_t^{\frac{t}{2\sqrt{u}}} \left(\int_{B(x,2t)} |(t^2L)^K r^2Le^{-r^2L}f(y)|^2 dy\right)^{\frac{1}{2}} \frac{dr}{r}\right)^2 \frac{dt}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\lesssim u^{-\frac{1}{4}}\left(\int_0^\infty \int_t^{\frac{t}{2\sqrt{u}}} \int_{B(x,2t)} |H_K(y,r)|^2\left(\frac{t}{r}\right)^{4K} dy\,\frac{dr}{r^2}\frac{dt}{t^n}\right)^{\frac{1}{2}}\\
&\lesssim u^{-\frac{1}{4}}\left(\int_0^\infty \int_{2\sqrt{u}r}^{r} \int_{B(x,2t)} |H_K(y,r)|^2\left(\frac{t}{r}\right)^{4K} dy\frac{dt}{t^n}\frac{dr}{r^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Applying Jensen's inequality to the integral in y and changing the variable t into rt we obtain

$$\begin{aligned}
F(x,u) &\lesssim u^{-\frac{1}{4}}\left(\int_0^\infty \int_{2\sqrt{u}r}^{r} \left(\int_{B(x,2t)} |H_K(y,r)|^{\widetilde{q}}dy\right)^{\frac{2}{\widetilde{q}}} \left(\frac{t}{r}\right)^{4K} \frac{dt}{t^{\frac{2n}{\widetilde{q}}}}\frac{dr}{r^2}\right)^{\frac{1}{2}}\\
&\lesssim u^{-\frac{1}{4}}\left(\int_0^\infty \int_{2\sqrt{u}}^{1} \left(\int_{B(x,2rt)} |H_K(y,r)|^{\widetilde{q}}dy\right)^{\frac{2}{\widetilde{q}}} t^{4K}\frac{dt}{t^{\frac{2n}{\widetilde{q}}}}\frac{dr}{r^{\frac{2n}{\widetilde{q}}+1}}\right)^{\frac{1}{2}}\\
&=: u^{-\frac{1}{4}}\widehat{H}(x,u).
\end{aligned}$$

Note that $1 < \frac{\widetilde{q}}{2}$. Then, applying Minkowski's integral inequality, and since for $\alpha := t \in (0,1)$ and $q := \frac{\widetilde{q}}{2}$ we can apply Proposition 2.61, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |F(x,u)|^2 w(x)dx &\lesssim u^{-\frac{1}{2}}\int_{\mathbb{R}^n}\widehat{H}(x,u)^2w(x)dx\\
&= u^{-\frac{1}{2}}\int_{2\sqrt{u}}^1\int_0^\infty\int_{\mathbb{R}^n}\left(\int_{B(x,2rt)}|H_K(y,r)|^{\widetilde{q}}dy\right)^{\frac{2}{\widetilde{q}}} w(x)dx\frac{dr}{r^{\frac{2n}{\widetilde{q}}+1}}t^{4K}\frac{dt}{t^{\frac{2n}{\widetilde{q}}}}\\
&\lesssim u^{-\frac{1}{2}}\int_{2\sqrt{u}}^1 t^{4K-\frac{2n}{\widetilde{q}}+\frac{n}{s}+1}\frac{dt}{t}\int_0^\infty\int_{\mathbb{R}^n}\left(\int_{B(x,2r)}|H_K(y,r)|^{\widetilde{q}}dy\right)^{\frac{2}{\widetilde{q}}} w(x)dx\frac{dr}{r^{\frac{2n}{\widetilde{q}}+1}}\\
&= u^{-\frac{1}{2}}u^{\frac{\widetilde{\theta}-1}{2}}\int_{\mathbb{R}^n}\widetilde{H}_K(x)^2w(x)dx, \qquad (3.21)
\end{aligned}$$

where in the last inequality we have used (3.20) and

$$\widetilde{H}_K(x) := \left(\int_0^\infty\left(\int_{B(x,2r)}|H_K(y,r)|^{\widetilde{q}}dy\right)^{\frac{2}{\widetilde{q}}}\frac{dr}{r^{\frac{2n}{\widetilde{q}}+1}}\right)^{\frac{1}{2}}.$$

Then, since $\{(r^2L)^{K+1}e^{-r^2L}\}_{r>0} \in \mathcal{F}_\infty(L^2 \to L^{\widetilde{q}})$ and $H_K(y,r) = 2^{K+1}e^{-\frac{r^2}{2}L}H_K\big(y,\frac{r}{\sqrt{2}}\big)$, applying Proposition 2.43 we get

$$\int_{\mathbb{R}^n}\widetilde{H}_K(x)^2w(x)dx \lesssim \sum_{j\geq 1}e^{-c4^j}\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2^{j+2}r)}\big|H_K\big(y,\tfrac{r}{\sqrt{2}}\big)\big|^2\frac{dy\,dr}{r^{n+1}}w(x)dx$$

$$
\begin{aligned}
&\lesssim \sum_{j\geq 1} e^{-c4^j} \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2^{j+2}\sqrt{2}r)} \left|H_K(y,r)\right|^2 \frac{dy\,dr}{r^{n+1}} w(x)dx \\
&\lesssim \sum_{j\geq 1} 2^{jnr} e^{-c4^j} \int_{\mathbb{R}^n} \mathcal{S}_{K+1,\mathrm{H}} f(x)^2 w(x)dx \\
&\lesssim \int_{\mathbb{R}^n} \mathcal{S}_{K+1,\mathrm{H}} f(x)^2 w(x)dx,
\end{aligned}
$$

where $r > r_w$ is so that $w \in A_r$ and $0 < 2 \leq 2r$. This, together with (3.21), yields

$$
I \lesssim \int_0^{\frac{1}{4}} u^{\frac{\widetilde{\theta}}{4}} \frac{du}{u} \|\mathcal{S}_{K+1,\mathrm{H}} f\|_{L^2(w)} \lesssim \|\mathcal{S}_{K+1,\mathrm{H}} f\|_{L^2(w)}.
$$

To estimate II we fix $\frac{1}{4} \leq u < \infty$ and observe that

$$
\left|(e^{-\frac{t^2}{4u}L} - e^{-t^2 L})f\right| \lesssim \int_{\frac{t}{2\sqrt{u}}}^{t} \left|r^2 L e^{-r^2 L} f\right| \frac{dr}{r}.
$$

Then, applying Minkowski's integral inequality, Jensen's inequality, Fubini, and the fact that we are integrating in $t < 2\sqrt{u}r$, we have

$$
\begin{aligned}
F(x,u) &\lesssim \left(\int_0^\infty \left(\int_{\frac{t}{2\sqrt{u}}}^{t} \left(\int_{B(x,2t)} |(t^2L)^K T_{r^2,0} f(y)|^2 dy \right)^{\frac{1}{2}} \frac{dr}{r} \right)^2 \frac{dt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^\infty \int_{\frac{t}{2\sqrt{u}}}^{t} \int_{B(x,2t)} |(t^2L)^K T_{r^2,0} f(y)|^2 dy \frac{dr}{r^2} \frac{dt}{t^n} \right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty \int_r^{2\sqrt{u}r} \int_{B(x,2t)} |(t^2L)^K T_{r^2,0} f(y)|^2 dy \frac{dt}{t^n} \frac{dr}{r^2} \right)^{\frac{p}{2}} \\
&\lesssim u^K \left(\int_0^\infty \int_r^{2\sqrt{u}r} \int_{B(x,4\sqrt{u}r)} |T_{r^2,K} f(y)|^2 dy\, dt \frac{dr}{r^{n+2}} \right)^{\frac{1}{2}} \\
&\lesssim u^{K+\frac{1}{4}} \left(\int_0^\infty \int_{B(x,4\sqrt{u}r)} |(r^2L)^{K+1} e^{-r^2L} f(y)|^2 dy \frac{dr}{r^{n+1}} \right)^{\frac{1}{2}}.
\end{aligned}
$$

Using this and Proposition 2.43, for $\alpha = 1$ and $\beta = 4\sqrt{u} > 1$, we get

$$
\begin{aligned}
II &\lesssim \int_{\frac{1}{4}}^\infty e^{-u} u^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |F(x,u)|^2 w(x)dx \right)^{\frac{1}{2}} \frac{du}{u} \\
&\lesssim \int_{\frac{1}{4}}^\infty e^{-u} u^{K+\frac{3}{4}} \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,4\sqrt{u}r)} |(r^2L)^{K+1} e^{-r^2L} f(y)|^2 \frac{dy\,dr}{r^{n+1}} w(x)dx \right)^{\frac{1}{2}} \frac{du}{u} \\
&\lesssim \int_{\frac{1}{4}}^\infty e^{-u} u^{K+\frac{3}{4}+\frac{nr}{4}} \frac{du}{u} \left(\int_{\mathbb{R}^n} |\mathcal{S}_{K+1,\mathrm{H}} f(x)|^2 w(x)dx \right)^{\frac{1}{2}} \lesssim \|\mathcal{S}_{K+1,\mathrm{H}} f\|_{L^2(w)},
\end{aligned}
$$

where $r > r_w$. □

Remark 3.22. *We note that Theorems 3.3 and 3.4 are restricted to functions $f \in L^2(\mathbb{R}^n)$. However, an inspection of the proof and a routine and tedious density argument allow us to extend these estimates to bigger classes* *of functions. For instance, we can take any function $f \in L^q(\widetilde{w})$ with $\widetilde{w} \in A_\infty$ and $q \in \mathcal{W}_{\widetilde{w}}(p_-(L), p_+(L))$. In that range the heat and the Poisson semigroups are uniformly bounded and satisfy off-diagonal estimates, hence the square functions under study are meaningfully defined. Moreover, L has a bounded holomorphic functional calculus on $L^q(\widetilde{w})$ (see [9], [10], and [11]). Further details are left to the interested reader.*

Next, using ideas from [55, Lemma 5.4], we obtain another comparison result. We will see its usefulness when proving some results in Section 3.3 and Chapter 4.

Theorem 3.23. *For all $w \in A_\infty$, and $f \in L^2(\mathbb{R}^n)$. There hold*

(*a*) $\|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathrm{G}_{m-1,\mathrm{H}} f\|_{L^p(w)}$, *for all $m \in \mathbb{N}$ and $0 < p < \infty$,*

(*b*) $\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathrm{G}_{K-1,\mathrm{P}} f\|_{L^p(w)}$, *for all $K \in \mathbb{N}$ and $0 < p < \infty$.*

Furthermore, one can see that (a) and (b) hold for all functions $f \in L^q(w)$ with $w \in A_\infty$ and $q \in \mathcal{W}_w(p_-(L), p_+(L))$.

Proof. We start by proving part (*a*). Fix $x \in \mathbb{R}^n$ and $t > 0$, and consider

$$B := B(x,t), \quad \widetilde{f}(y) := (t^2L)^{m-1} e^{-\frac{t^2}{2}L} f(y), \quad \text{and} \quad H(y) := \widetilde{f}(y) - (\widetilde{f})_{4B},$$

where $(\widetilde{f})_{4B} = \fint_{4B} \widetilde{f}(y)\, dy$. Then, applying the fact that $\{t^2Le^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^2)$ and that $t^2Le^{-\frac{t^2}{2}L}1 = t^2L1 = 0$ (see [3]), we obtain that

$$\begin{aligned}
\left(\int_B |t^2Le^{-\frac{t^2}{2}L}\widetilde{f}(y)|^2\, dy\right)^{\frac12} &= \left(\int_B |t^2Le^{-\frac{t^2}{2}L}H(y)|^2\, dy\right)^{\frac12} \\
&\lesssim \left(\int_B |t^2Le^{-\frac{t^2}{2}L}(H\mathbf{1}_{4B})(y)|^2\, dy\right)^{\frac12} + \sum_{j\geq 2}\left(\int_B |t^2Le^{-\frac{t^2}{2}L}(H\mathbf{1}_{C_j(B)})(y)|^2\, dy\right)^{\frac12} \\
&\lesssim \left(\int_{4B} |H(y)|^2\, dy\right)^{\frac12} + \sum_{j\geq 2} e^{-c4^j}\left(\int_{2^{j+1}B} |H(y)|^2\, dy\right)^{\frac12} =: I + \sum_{j\geq 2} e^{-c4^j} I_j.
\end{aligned}$$

By Poincaré inequality, we conclude that

$$I \lesssim t \left(\int_{8B} |\nabla \widetilde{f}(y)|^2\, dy\right)^{\frac12},$$

and that

$$\begin{aligned}
I_j &\lesssim \left(\int_{2^{j+1}B} |\widetilde{f}(y) - (\widetilde{f})_{2^{j+1}B}|^2\, dy\right)^{\frac12} + |2^{j+1}B|^{1/2} \sum_{k=2}^{j} |(\widetilde{f})_{2^kB} - (\widetilde{f})_{2^{k+1}B}| \\
&\lesssim |2^{j+1}B|^{1/2} \sum_{k=2}^{j} \left(\fint_{2^{k+1}B} |\widetilde{f}(y) - (\widetilde{f})_{2^{k+1}B}|^2\, dy\right)^{\frac12} \lesssim \sum_{k=2}^{j} 2^{(j-k)n/2} 2^k t \left(\int_{2^{k+2}B} |\nabla \widetilde{f}(y)|^2\, dy\right)^{\frac12}.
\end{aligned}$$

Then,

$$\left(\int_B |t^2Le^{-\frac{t^2}{2}L}\widetilde{f}(y)|^2\, dy\right)^{\frac12} \lesssim t\left(\int_{8B} |\nabla \widetilde{f}(y)|^2 dy\right)^{\frac12} + \sum_{j\geq 2} e^{-c4^j} \sum_{k=2}^{j} 2^{(j-k)n/2} 2^k t \left(\int_{2^{k+2}B} |\nabla \widetilde{f}(y)|^2\, dy\right)^{\frac12}$$

$$\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_{2^{j+2}B} |t\nabla \widetilde{f}(y)|^2 \, dy \right)^{\frac{1}{2}},$$

and therefore,

$$\mathcal{S}_{m,\mathrm{H}} f(x) \lesssim \sum_{j\geq 1} e^{-c4^j} \mathrm{G}_{m-1,\mathrm{H}}^{2^{j+3}} f(x),$$

recall the definition of $\mathrm{G}_{m-1,\mathrm{H}}^{2^{j+3}}$ in (1.25) and (1.27).

Next, for every $0 < p < \infty$ and $w \in A_\infty$, choosing $\widehat{r} > \max\{r_w, p/2\}$, taking the $L^p(w)$ norm in both sides of the previous inequality, and applying change of angles (see Proposition 2.43), we conclude that

$$\|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \sum_{j\geq 1} e^{-c4^j} \|\mathrm{G}_{m-1,\mathrm{H}}^{2^{j+3}} f\|_{L^p(w)} \lesssim \|\mathrm{G}_{m-1,\mathrm{H}} f\|_{L^p(w)} \sum_{j\geq 1} 2^{jn\widehat{r}/p} e^{-c4^j} \lesssim \|\mathrm{G}_{m-1,\mathrm{H}} f\|_{L^p(w)}.$$

As for part (b), fix $w \in A_\infty$, $f \in L^2(\mathbb{R}^n)$, and $0 < p < \infty$, and note that following the same argument of [55, Lemma 5.4][1], there exists a dimensional constant $k_0 \in \mathbb{N}$ and $C_1 > 0$ such that for all $K \in \mathbb{N}$ and $k \in N_0$.

$$\mathcal{S}_{K,\mathrm{P}}^{2^k} f(x) \leq C_1 \left(\mathrm{G}_{K-1,\mathrm{P}}^{2^{k+k_0}} f(x) \right)^{\frac{1}{2}} \left(\mathcal{S}_{K,\mathrm{P}}^{2^{k+k_0}} f(x) \right)^{\frac{1}{2}},$$

recall the definitions of $\mathcal{S}_{K,\mathrm{P}}^{\alpha}$ and $\mathrm{G}_{K-1,\mathrm{P}}^{\alpha}$, for $\alpha > 0$, in (1.25), (1.29), and (1.30). Now, for some $R > 0$, to be determined later, consider

$$\mathcal{S}^* f(x) := \sum_{k=0}^{\infty} R^{-k} \mathcal{S}_{K,\mathrm{P}}^{2^k} f(x) \quad \text{and} \quad \mathrm{G}^* f(x) := \sum_{k=0}^{\infty} R^{-k} \mathrm{G}_{K-1,\mathrm{P}}^{2^k} f(x).$$

By the above inequality, and using Young's inequality, we have

$$\begin{aligned} \mathcal{S}^* f(x) &\leq \sum_{k=0}^{\infty} R^{-(k+k_0)} \left(C_1^2 R^{2k_0} \mathrm{G}_{K-1,\mathrm{P}}^{2^{k+k_0}} f(x) \right)^{\frac{1}{2}} \left(\mathcal{S}_{K,\mathrm{P}}^{2^{k+k_0}} f(x) \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(C_1^2 R^{2k_0} \sum_{k=0}^{\infty} R^{-(k+k_0)} \mathrm{G}_{K-1,\mathrm{P}}^{2^{k+k_0}} f(x) + \sum_{k=0}^{\infty} R^{-(k+k_0)} \mathcal{S}_{K,\mathrm{P}}^{2^{k+k_0}} f(x) \right) \\ &\leq \frac{1}{2} \left(R^{2k_0} C_1^2 \mathrm{G}^* f(x) + \mathcal{S}^* f(x) \right). \end{aligned} \tag{3.24}$$

Besides, since $\mathcal{S}_{K,\mathrm{P}}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ (see Theorem 3.2), applying [32, Proposition 4, Section 3] or [4], and choosing $R > 2^{\frac{n}{2}+1}$, we have that

$$\|\mathcal{S}^* f\|_{L^2(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty} R^{-k} \|\mathcal{S}_{K,\mathrm{P}}^{2^k} f\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} R^{-k} 2^{\frac{kn}{2}} \|\mathcal{S}_{K,\mathrm{P}} f\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} R^{-k} 2^{\frac{kn}{2}} \|f\|_{L^2(\mathbb{R}^n)} < \infty,$$

hence $\mathcal{S}^* f(x) < \infty$ a. e. $x \in \mathbb{R}^n$. Then, by (3.24),

$$\mathcal{S}_{K,\mathrm{P}} f(x) \leq \mathcal{S}^* f(x) \leq C R^{2k_0} \mathrm{G}^* f(x).$$

Hence, taking the $L^p(w)$ norm in the previous inequality, by Proposition 2.60, we conclude that, for $r_0 > \max\{p/2, r_w\}$ and $R = 2^{\frac{nr_0}{p}+1} > 2^{\frac{n}{2}+1}$,

$$\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \sum_{k=0}^{\infty} R^{-(k-2k_0)} \|\mathrm{G}_{K-1,\mathrm{P}}^{2^k} f\|_{L^p(w)} \lesssim R^{2k_0} \sum_{k=0}^{\infty} R^{-k} 2^{\frac{knr_0}{p}} \|\mathrm{G}_{K-1,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathrm{G}_{K-1,\mathrm{P}} f\|_{L^p(w)}.$$

Following the explanation of Remark 3.22 we conclude (a) and (b) for all functions $f \in L^q(w)$ with $w \in A_\infty$ and $q \in \mathcal{W}_w(p_-(L), p_+(L))$. □

[1]We want to thank Steve Hofmann for sharing with us this argument that was omitted in [55, Lemma 5.4].

3.2 Non-tangential maximal functions

We first show boundedness on $L^p(w)$ for the non-tangential maximal functions in (1.32).

Theorem 3.25. *Given $w \in A_\infty$. There hold*

(*a*) $\mathcal{N}_{\mathrm{H}}$ *is bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), \infty)$,*

(*b*) $\mathcal{N}_{\mathrm{P}}$ *is bounded on $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), p_+(L))$.*

Proof. We start by proving part (*a*). Fix $w \in A_\infty$, $p \in \mathcal{W}_w(p_-(L), \infty)$, and $f \in L^\infty_c(\mathbb{R}^n)$. Take $p_0 \in (p_-(L), 2)$ and apply the $L^{p_0}(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimates satisfied by the family $\{e^{-t^2L}\}_{t>0}$ to obtain

$$\begin{aligned}
\|\mathcal{N}_{\mathrm{H}} f\|_{L^p(w)} &\le \left(\int_{\mathbb{R}^n} \sup_{t>0} \left(\int_{B(x,2t)} |e^{-t^2L} f(z)|^2 \frac{dz}{t^n} \right)^{\frac{p}{2}} w(x)dx \right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\ge 1} e^{-c4^j} \left(\int_{\mathbb{R}^n} \sup_{t>0} \left(\int_{B(x,2^{j+2}t)} |f(z)|^{p_0} \frac{dz}{t^n} \right)^{\frac{p}{p_0}} w(x)dx \right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\ge 1} e^{-c4^j} 2^{\frac{jn}{p_0}} \left(\int_{\mathbb{R}^n} |\mathcal{M}_{p_0} f(x)|^p w(x)dx \right)^{\frac{1}{p}} \\
&\lesssim \|\mathcal{M}_{p_0} f\|_{L^p(w)},
\end{aligned}$$

where $\mathcal{M}_{p_0} f := (\mathcal{M}|f|^{p_0})^{\frac{1}{p_0}}$.

Now, take $p_-(L) < p_0 < 2$ close enough to $p_-(L)$ so that $w \in A_{\frac{p}{p_0}}$. Consequently, $\mathcal{M}_{p_0}$ is bounded on $L^p(w)$, and then, we conclude that

$$\|\mathcal{N}_{\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathcal{M}_{p_0} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},$$

for all $f \in L^\infty_c(\mathbb{R}^n)$, and by a density argument, for all $f \in L^p(w)$.

We next prove part (*b*). In this case, we fix $w \in A_\infty$, $p \in \mathcal{W}_w(p_-(L), p_+(L))$, and $f \in L^\infty_c(\mathbb{R}^n)$. First, notice that we can split $\mathcal{N}_{\mathrm{P}}$ as follows

$$\begin{aligned}
\mathcal{N}_{\mathrm{P}} f(x) &\le \sup_{(y,t)\in\Gamma(x)} \left(\int_{B(y,t)} |(e^{-t\sqrt{L}} - e^{-t^2L}) f(z)|^2 \frac{dz}{t^n} \right)^{\frac{1}{2}} + \sup_{(y,t)\in\Gamma(x)} \left(\int_{B(y,t)} |e^{-t^2L} f(z)|^2 \frac{dz}{t^n} \right)^{\frac{1}{2}} \\
&= \sup_{(y,t)\in\Gamma(x)} \left(\int_{B(y,t)} |(e^{-t\sqrt{L}} - e^{-t^2L}) f(z)|^2 \frac{dz}{t^n} \right)^{\frac{1}{2}} + \mathcal{N}_{\mathrm{H}} f(x) =: \mathfrak{m}_{\mathrm{P}} f(x) + \mathcal{N}_{\mathrm{H}} f(x).
\end{aligned}$$

After applying subordination formula (1.36) and Minkowski's integral inequality, we obtain that

$$\begin{aligned}
\mathfrak{m}_{\mathrm{P}} f(x) &\lesssim \sup_{t>0} \int_0^\infty e^{-u} u^{\frac{1}{2}} \left(\int_{B(x,2t)} |(e^{-\frac{t^2}{4u}L} - e^{-t^2L}) f(y)|^2 \frac{dy}{t^n} \right)^{\frac{1}{2}} \frac{du}{u} \\
&\lesssim \sup_{t>0} \int_0^{\frac{1}{4}} u^{\frac{1}{2}} \left(\int_{B(x,2t)} |(e^{-\frac{t^2}{4u}L} - e^{-t^2L}) f(y)|^2 \frac{dy}{t^n} \right)^{\frac{1}{2}} \frac{du}{u} \\
&\quad + \sup_{t>0} \int_{\frac{1}{4}}^\infty e^{-u} u^{\frac{1}{2}} \left(\int_{B(x,2t)} |(e^{-\frac{t^2}{4u}L} - e^{-t^2L}) f(y)|^2 \frac{dy}{t^n} \right)^{\frac{1}{2}} \frac{du}{u} =: I + II.
\end{aligned}$$

We first deal with I. Take $p_-(L) < p_0 < 2$, and apply the $L^{p_0}(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimates satisfied by $\{e^{-t^2L}\}_{t>0}$. Then,

$$I = \sup_{t>0} \int_0^{\frac{1}{4}} u^{\frac{1}{2}} \left(\int_{B(x,2t)} |e^{-\frac{t^2}{2}L}(e^{-(\frac{1}{4u}-\frac{1}{2})t^2L} - e^{-\frac{t^2}{2}L}) f(y)|^2 \frac{dy}{t^n} \right)^{\frac{1}{2}} \frac{du}{u}$$
$$\lesssim \sup_{t>0} \sum_{j\geq 1} e^{-c4^j} \int_0^{\frac{1}{4}} u^{\frac{1}{2}} \left(\fint_{B(x,2^{j+2}t)} |(e^{-(\frac{1}{4u}-\frac{1}{2})t^2L} - e^{-\frac{t^2}{2}L}) f(y)|^{p_0} dy \right)^{\frac{1}{p_0}} \frac{du}{u}.$$

Now, notice that when $0 < u < \frac{1}{4}$, we have

$$\left| \left(e^{-(\frac{1}{4u}-\frac{1}{2})t^2L} - e^{-\frac{t^2}{2}L} \right) f(y) \right| \leq 2 \int_{\frac{t}{\sqrt{2}}}^{t\sqrt{\frac{1}{4u}-\frac{1}{2}}} |r^2Le^{-r^2L} f(y)| \frac{dr}{r} \leq 2 \int_{\frac{t}{\sqrt{2}}}^{\frac{t}{2\sqrt{u}}} |r^2Le^{-r^2L} f(y)| \frac{dr}{r}$$
$$\lesssim \log(u^{-\frac{1}{2}})^{\frac{1}{2}} \left(\int_0^\infty |r^2Le^{-r^2L} f(y)|^2 \frac{dr}{r} \right)^{\frac{1}{2}} =: \log(u^{-\frac{1}{2}})^{\frac{1}{2}}\, g_{\mathrm{H}} f(y).$$

Therefore,

$$I \lesssim \sum_{j\geq 1} e^{-c4^j} \int_0^{\frac{1}{4}} \log(u^{-\frac{1}{2}})^{\frac{1}{2}} u^{\frac{1}{2}} \frac{du}{u} \sup_{t>0} \left(\fint_{B(x,2^{j+2}t)} |g_{\mathrm{H}} f(y)|^{p_0} dy \right)^{\frac{1}{p_0}} \lesssim \mathcal{M}_{p_0}(g_{\mathrm{H}} f)(x).$$

On the other hand, for $\frac{1}{4} \leq u < \infty$,

$$\left| \left(e^{-\frac{t^2}{4u}L} - e^{-t^2L} \right) f(y) \right| \leq 2 \int_{\frac{t}{2\sqrt{u}}}^{t} |r^2Le^{-r^2L} f(y)| \frac{dr}{r} \lesssim \log(2\sqrt{u})^{\frac{1}{2}} \left(\int_{\frac{t}{2\sqrt{u}}}^{t} |r^2Le^{-r^2L} f(y)|^2 \frac{dr}{r} \right)^{\frac{1}{2}}.$$

Hence,

$$II \lesssim \sup_{t>0} \int_{\frac{1}{4}}^\infty e^{-u} \log(2\sqrt{u})^{\frac{1}{2}} u^{\frac{1}{2}} \left(\int_{B(x,2t)} \int_{\frac{t}{2\sqrt{u}}}^{t} |r^2Le^{-r^2L} f(y)|^2 \frac{dr}{r} \frac{dy}{t^n} \right)^{\frac{1}{2}} \frac{du}{u}$$
$$\lesssim \sup_{t>0} \int_{\frac{1}{4}}^\infty u e^{-u} \left(\int_{\frac{t}{2\sqrt{u}}}^{t} \int_{B(x,2t)} |r^2Le^{-r^2L} f(y)|^2 \frac{dy}{t^n} \frac{dr}{r} \right)^{\frac{1}{2}} \frac{du}{u}$$
$$\lesssim \sup_{t>0} \int_{\frac{1}{4}}^\infty e^{-u} \left(\int_{\frac{t}{2\sqrt{u}}}^{t} \int_{B(x,4\sqrt{u}r)} |r^2Le^{-r^2L} f(y)|^2 \frac{dy\,dr}{r^{n+1}} \right)^{\frac{1}{2}} du$$
$$\lesssim \int_{\frac{1}{4}}^\infty e^{-u} \left(\int_0^\infty \int_{B(x,4\sqrt{u}r)} |r^2Le^{-r^2L} f(y)|^2 \frac{dy\,dr}{r^{n+1}} \right)^{\frac{1}{2}} du = \int_{\frac{1}{4}}^\infty e^{-u}\, \mathcal{S}_{\mathrm{H}}^{4\sqrt{u}} f(x) du,$$

recall the definition of $\mathcal{S}_{\mathrm{H}}^{4\sqrt{u}}$ in (1.25) and (1.26). Gathering these estimates gives us, for $p_-(L) < p_0 < 2$,

$$\mathcal{N}_{\mathrm{P}} f(x) \lesssim \mathcal{M}_{p_0}(g_{\mathrm{H}} f)(x) + \int_{\frac{1}{4}}^\infty e^{-u}\, \mathcal{S}_{\mathrm{H}}^{4\sqrt{u}} f(x)\, du + \mathcal{N}_{\mathrm{H}} f(x), \quad \forall x \in \mathbb{R}^n.$$

Let $w \in A_\infty$ and $p \in \mathcal{W}_w(p_-(L), p_+(L))$, taking norms on $L^p(w)$, and applying Proposition 2.60, we obtain, for $r > \max\{p/2, r_w\}$ and $p_-(L) < p_0 < 2$,

$$\|\mathcal{N}_{\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{M}_{p_0}(g_{\mathrm{H}} f)\|_{L^p(w)} + \int_{\frac{1}{4}}^\infty u^{\frac{nr}{2p}} e^{-u}\, du\, \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)} + \|\mathcal{N}_{\mathrm{H}} f\|_{L^p(w)}$$
$$\lesssim \|\mathcal{M}_{p_0}(g_{\mathrm{H}} f)\|_{L^p(w)} + \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)} + \|\mathcal{N}_{\mathrm{H}} f\|_{L^p(w)}.$$

Now, taking p_0 close enough to $p_-(L)$ so that $w \in A_{\frac{p}{p_0}}$, we have that the maximal operator $\mathcal{M}_{p_0}$ is bounded on $L^p(w)$. Besides, since $p \in \mathcal{W}_w(p_-(L), p_+(L)) \subset \mathcal{W}_w(p_-(L), \infty)$, we have that g_{H}, $\mathcal{S}_{\mathrm{H}}$, and $\mathcal{N}_{\mathrm{H}}$ are bounded operators on $L^p(w)$, (see [11, Theorem 7.6, (*a*)], Theorem 3.1, part (*a*), and Theorem 3.25, part (*a*), respectively), we conclude (*b*) for all $f \in L_c^\infty(\mathbb{R}^n)$, and by a density argument, for all $f \in L^p(w)$. $\square$

We next compare the norms of $\mathcal{N}_{\mathrm{H}}$ and $\mathcal{N}_{\mathrm{P}}$, with the norms of G_{H} and $\mathcal{G}_{\mathrm{P}}$, respectively. To this end, we shall need the following Lemma which is related with the change of angle and whose proof follows similarly to the proof of [55, Lemma 6.2]. Consider for all $\kappa \geq 1$

$$\mathcal{N}^\kappa f(x) := \sup_{(y,t)\in\Gamma^\kappa(x)} \left(\int_{B(y,\kappa t)} |F(z,t)|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}},$$

we simply write $\mathcal{N}$ for $\kappa = 1$.

Lemma 3.26. *Let $w \in A_r$, $0 < p < \infty$, and $\kappa \geq 1$. There holds*

$$\|\mathcal{N}^\kappa f\|_{L^p(w)} \lesssim \kappa^{n\left(\frac{1}{2}+\frac{r}{p}\right)} \|\mathcal{N} f\|_{L^p(w)}.$$

Proof. Consider $O_\lambda := \{x \in \mathbb{R}^n : \mathcal{N} f(x) > \lambda\}$, $E_\lambda := \mathbb{R}^n \backslash O_\lambda$, and, for $\gamma = 1 - \frac{1}{(4\kappa)^n}$, the set of γ-density $E_\lambda^* := \left\{x \in \mathbb{R}^n : \forall\, r > 0,\ \frac{|E_\lambda \cap B(x,r)|}{|B(x,r)|} \geq \gamma\right\}$. Note that $O_\lambda^* := \mathbb{R}^n \setminus E_\lambda^* = \{x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_\lambda})(x) > 1/(4\kappa)^n\}$.

We claim that for every $\lambda > 0$,

$$\mathcal{N}^\kappa f(x) \leq (3\kappa)^{\frac{n}{2}} \lambda, \quad \text{for all} \quad x \in E_\lambda^*. \tag{3.27}$$

Assuming this, let $0 < p < \infty$ and $w \in A_r$, $1 \leq r < \infty$, then $\mathcal{M} : L^r(w) \to L^{r,\infty}(w)$. Hence, we would have

$$\begin{aligned}
\|\mathcal{N}^\kappa f\|_{L^p(w)}^p &= p \int_0^\infty \lambda^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{N}^\kappa f(x) > \lambda\})\, d\lambda \\
&= p(3\kappa)^{\frac{np}{2}} \int_0^\infty \lambda^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{N}^\kappa f(x) > (3\kappa)^{\frac{n}{2}}\lambda\})\, d\lambda \leq p(3\kappa)^{\frac{np}{2}} \int_0^\infty \lambda^{p-1} w(O_\lambda^*)\, d\lambda \\
&\lesssim p(3\kappa)^{\frac{np}{2}} (4k)^{nr} \int_0^\infty \lambda^{p-1} w(O_\lambda)\, d\lambda = (3\kappa)^{\frac{np}{2}} (4k)^{nr} \|\mathcal{N} f\|_{L^p(w)}^p,
\end{aligned}$$

which would finish the proof.

So it just remains to show (3.27). First, note that if $x \in E_\lambda^*$ then, for every $(y,t) \in \Gamma^{2\kappa}(x)$, $B(y,t) \cap E_\lambda \neq \emptyset$. To prove this, suppose by way of contradiction that $B(y,t) \subset O_\lambda$. Then, since $B(y,t) \subset B(x, 3\kappa t)$,

$$\mathcal{M}(\mathbf{1}_{O_\lambda})(x) \geq \frac{|B(y,t)|}{|B(x,3\kappa t)|} = \frac{1}{(3\kappa)^n} > \frac{1}{(4\kappa)^n},$$

which implies that $x \in O_\lambda^*$, a contradiction. Therefore, for all $(y,t) \in \Gamma^{2\kappa}(x)$, with $x \in E_\lambda^*$, there exists $y_0 \in B(y,t)$ (in particular $(y,t) \in \Gamma(y_0)$) such that $\mathcal{N} f(y_0) \leq \lambda$. Hence, for all $(y,t) \in \Gamma^{2\kappa}(x)$, with $x \in E_\lambda^*$,

$$\left(\int_{B(y,t)} |F(\xi,t)|^2 \frac{d\xi}{t^n}\right)^{\frac{1}{2}} \leq \sup_{(z,s)\in\Gamma(y_0)} \left(\int_{B(z,s)} |F(\xi,s)|^2 \frac{d\xi}{s^n}\right)^{\frac{1}{2}} \leq \lambda. \tag{3.28}$$

On the other hand, for all $x \in \mathbb{R}^n$ and $(y,t) \in \Gamma^\kappa(x)$, we have that $B(y,\kappa t) \subset \bigcup_i B(y_i,t)$, where $\{B(y_i,t)\}_i$ is a collection of at most $(3\kappa)^n$ balls such that $y_i \in B(x, 2\kappa t)$, or equivalently $(y_i,t) \in \Gamma^{2\kappa}(x)$.

Therefore, we conclude that, for all $x \in E_\lambda^*$ and $(y,t) \in \Gamma^\kappa(x)$

$$\int_{B(y,\kappa t)} |F(z,t)|^2 \frac{dz}{t^n} \leq \sum_i \int_{B(y_i,t)} |F(z,t)|^2 \frac{dz}{t^n} \leq (3\kappa)^n \lambda^2,$$

where we have used (3.28), since $x \in E_\lambda^*$ and $(y_i, t) \in \Gamma^{2\kappa}(x)$. Finally taking the supremum over all $(y, t) \in \Gamma^\kappa(x)$, we obtain

$$\mathcal{N}^\kappa f(x)^2 \leq (3\kappa)^n \lambda^2, \quad \forall\, x \in E_\lambda^*.$$

Raising both sides of the previous inequality to $\frac{1}{2}$ we conclude (3.27) and the proof.

□

Theorem 3.29. *Given an arbitrary $f \in L^2(\mathbb{R}^n)$, for all $w \in A_\infty$ and $0 < p < \infty$, there hold*

(*a*) $\|\mathcal{G}_{\rm P} f\|_{L^p(w)} \lesssim \|\mathcal{N}_{\rm P} f\|_{L^p(w)}$,

(*b*) $\|{\rm G}_{\rm H} f\|_{L^p(w)} \lesssim \|\mathcal{N}_{\rm H} f\|_{L^p(w)}$.

Proof. We start by proving part (*a*). Fix $w \in A_\infty$, $0 < p < \infty$, and $f \in L^2(\mathbb{R}^n)$. For every $N > 1$ and $\alpha \geq 1$, we define

$$K_N := \{(y, t) \in \mathbb{R}^{n+1}_+ : y \in B(0, N), t \in (N^{-1}, N)\} \tag{3.30}$$

and

$$\mathcal{G}^\alpha_{{\rm P},N} f(x) := \left(\int_0^\infty \int_{B(x,\alpha t)} \mathbf{1}_{K_N}(y, t) |t\nabla_{y,t} e^{-t\sqrt{L}} f(y)|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

when $\alpha = 1$ we simply write $\mathcal{G}_{{\rm P},N}$. Then, $\operatorname{supp} \mathcal{G}^\alpha_{{\rm P},N} f \subset B(0, (\alpha + 1)N)$ and, since the vertical square function $\left(\int_0^\infty |t\nabla_{y,t} e^{-t\sqrt{L}} f(y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^n)$, we have that $\|\mathcal{G}^\alpha_{{\rm P},N} f\|_{L^p(w)} \leq CN^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)} w(B(0, (\alpha + 1)N))^{\frac{1}{p}} < \infty$.

Following the ideas used in the proofs of [55, Theorems 6.1 and 7.1], for every $\lambda > 0$, set

$$O_\lambda := \{x \in \mathbb{R}^n : \mathcal{N}^\kappa_{\rm P} f(x) > \lambda\} \quad \text{and} \quad E_\lambda := \mathbb{R}^n \setminus O_\lambda,$$

where

$$\mathcal{N}^\kappa_{\rm P} f(x) = \sup_{(y,t) \in \Gamma^\kappa(x)} \left(\int_{B(y,\kappa t)} |e^{-t\sqrt{L}} f(z)|^2 \frac{dz}{t^n} \right)^{\frac{1}{2}},$$

and κ is some positive number that we will determine during the proof.

Besides, consider the set of $\frac{1}{2}$-density associated with E_λ:

$$E_\lambda^* := \left\{ x \in \mathbb{R}^n : \forall r > 0, \frac{|E_\lambda \cap B(x, r)|}{|B(x, r)|} \geq \frac{1}{2} \right\} \text{ and } O_\lambda^* := \mathbb{R}^n \setminus E_\lambda^* = \left\{ x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_\lambda})(x) > \frac{1}{2} \right\}.$$

We have that $E_\lambda^* \subset E_\lambda$ and $O_\lambda \subset O_\lambda^*$. Moreover, since $w \in A_\infty$, $\mathcal{M} : L^r(w) \to L^{r,\infty}(w)$, for some fixed $r > r_w$. Consequently $w(O_\lambda^*) \leq C_w w(O_\lambda)$. On the other hand, consider the set

$$\widetilde{O}_\lambda := \{x \in \mathbb{R}^n : \mathcal{G}^\alpha_{{\rm P},N} f(x) > \lambda\}.$$

Proceeding as in the proof of Proposition 2.43, part (*a*), we can show that $\widetilde{O}_\lambda$ is open and, since $\|\mathcal{G}^\alpha_{{\rm P},N} f\|_{L^p(w)} < \infty$, then $w(\widetilde{O}_\lambda) < \infty$ which implies that $\widetilde{O}_\lambda \subsetneq \mathbb{R}^n$. Hence, taking a Whitney decomposition of $\widetilde{O}_\lambda$, there exists a family of closed cubes $\{Q_j\}_{j \in \mathbb{N}}$ with disjoint interiors such that

$$\bigcup_{j \in \mathbb{N}} Q_j = \widetilde{O}_\lambda \quad \text{and} \quad \operatorname{diam}(Q_j) \leq d(Q_j, \mathbb{R}^n \setminus \widetilde{O}_\lambda) \leq 4 \operatorname{diam}(Q_j).$$

We claim that there exists a positive constant c_w, depending on the weight, such that, for every $0 < \gamma < 1$ and $\alpha = 12\sqrt{n}$,

$$w(\{x \in E^*_{\gamma\lambda} : \mathcal{G}_{\mathrm{P},N} f(x) > 2\lambda, \mathcal{N}^\kappa_{\mathrm{P}} f(x) \le \gamma\lambda\}) \le C\gamma^{c_w} w(\{x \in \mathbb{R}^n : \mathcal{G}^\alpha_{\mathrm{P},N} f(x) > \lambda\}). \tag{3.31}$$

Assuming this momentarily, we would have

$$\begin{aligned} w(\{x \in \mathbb{R}^n : \mathcal{G}_{\mathrm{P},N} f(x) > 2\lambda\}) &\le w(O^*_{\gamma\lambda}) + w(\{x \in E^*_{\gamma\lambda} : \mathcal{G}_{\mathrm{P},N} f(x) > 2\lambda, \mathcal{N}^\kappa_{\mathrm{P}} f(x) \le \gamma\lambda\}) + w(O_{\gamma\lambda}) \\ &\le C\gamma^{c_w} w(\{x \in \mathbb{R}^n : \mathcal{G}^\alpha_{\mathrm{P},N} f(x) > \lambda\}) + Cw(\{x \in \mathbb{R}^n : \mathcal{N}^\kappa_{\mathrm{P}} f(x) > \gamma\lambda\}). \end{aligned}$$

Multiplying both sides of the previous inequality by λ^{p-1} and integrating in $\lambda > 0$, we would obtain

$$\|\mathcal{G}_{\mathrm{P},N} f\|^p_{L^p(w)} \le C\gamma^{c_w} \|\mathcal{G}^\alpha_{\mathrm{P},N} f\|^p_{L^p(w)} + C_\gamma \|\mathcal{N}^\kappa_{\mathrm{P}} f\|^p_{L^p(w)}.$$

Then, applying Proposition 2.43 and Lemma 3.26 with $\mathcal{N} = \mathcal{N}_{\mathrm{P}}$ we would get

$$\|\mathcal{G}_{\mathrm{P},N} f\|^p_{L^p(w)} \le C_\alpha\gamma^{c_w} \|\mathcal{G}_{\mathrm{P},N} f\|^p_{L^p(w)} + C_{\kappa,\gamma} \|\mathcal{N}_{\mathrm{P}} f\|^p_{L^p(w)}.$$

Finally, since $\|\mathcal{G}_{\mathrm{P},N} f\|_{L^p(w)} \le \|\mathcal{G}^\alpha_{\mathrm{P},N} f\|_{L^p(w)} < \infty$, taking γ small enough such that $C_\alpha\gamma^{c_w} < \frac{1}{2}$, we would conclude that, for some constant $C > 0$ uniform on N,

$$\|\mathcal{G}_{\mathrm{P},N} f\|_{L^p(w)} \le C\|\mathcal{N}_{\mathrm{P}} f\|_{L^p(w)}.$$

This and the Monotone Convergence Theorem would readily lead to the desired estimate. Therefore to complete the proof we just need to show (3.31). Notice that since $\mathcal{G}_{\mathrm{P},N} f \le \mathcal{G}^\alpha_{\mathrm{P},N} f$, we have

$$\left\{x \in E^*_{\gamma\lambda} : \mathcal{G}_{\mathrm{P},N} f(x) > 2\lambda, \mathcal{N}^\kappa_{\mathrm{P}} f(x) \le \gamma\lambda\right\} \subset \bigcup_{j\in\mathbb{N}} \left\{x \in E^*_{\gamma\lambda} \cap Q_j : \mathcal{G}_{\mathrm{P},N} f(x) > 2\lambda, \mathcal{N}^\kappa_{\mathrm{P}} f(x) \le \gamma\lambda\right\}.$$

Consequently, since $w \in A_\infty$, to obtain (3.31) it is enough to show

$$|\{x \in E^*_{\gamma\lambda} \cap Q_j : \mathcal{G}_{\mathrm{P},N} f(x) > 2\lambda, \mathcal{N}^\kappa_{\mathrm{P}} f(x) \le \gamma\lambda\}| \le C\gamma^2 |Q_j|. \tag{3.32}$$

To this end, consider $u(y,t) = e^{-t\sqrt{L}} f(y)$, and

$$\mathcal{G}_{\mathrm{P},1,j,N} f(x) := \left(\int_{\frac{\ell(Q_j)}{2}}^\infty \int_{B(x,t)} \mathbf{1}_{K_N}(y,t) \left|t\nabla_{y,t} u(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$\text{and}\quad \mathcal{G}_{\mathrm{P},2,j,N} f(x) := \left(\int_0^{\frac{\ell(Q_j)}{2}} \int_{B(x,t)} \mathbf{1}_{K_N}(y,t) \left|t\nabla_{y,t} u(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

We have that $\mathcal{G}_{\mathrm{P},N} f \le \mathcal{G}_{\mathrm{P},1,j,N} f + \mathcal{G}_{\mathrm{P},2,j,N} f$ and that $\mathcal{G}_{\mathrm{P},1,j,N} f(x) \le \lambda$ for all $x \in Q_j$. Indeed, notice that for each j, there exists $x_j \in \mathbb{R}^n \setminus \widetilde{O}_\lambda$ such that $d(x_j, Q_j) \le 4\mathrm{diam} Q_j$. Besides, if (y,t) is such that $t \ge \frac{\ell(Q_j)}{2}$, $x \in Q_j$, and $y \in B(x,t)$, then

$$|x_j - y| \le |x_j - x| + |x - y| < 5\sqrt{n}\ell(Q_j) + t \le 11\sqrt{n}t.$$

Hence, for $\alpha = 12\sqrt{n}$ and for all $x \in Q_j$, we have

$$\begin{aligned} \mathcal{G}_{\mathrm{P},1,j,N} f(x)^2 &= \int_{\frac{\ell(Q_j)}{2}}^\infty \int_{B(x,t)} \mathbf{1}_{K_N}(y,t) \left|t\nabla_{y,t} u(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}} \\ &\le \iint_{\Gamma^\alpha(x_j)} \mathbf{1}_{K_N}(y,t) \left|t\nabla_{y,t} u(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}} = \mathcal{G}^\alpha_{\mathrm{P},N} f(x_j)^2 \le \lambda^2. \end{aligned}$$

This and Chebychev's inequality imply that

$$
\begin{aligned}
&|\{x \in E^*_{\gamma\lambda} \cap Q_j : \mathcal{G}_{\mathrm{P},N} f(x) > 2\lambda, \mathcal{N}^\kappa_{\mathrm{P}} f(x) \le \gamma\lambda\}| \\
&\qquad \le |\{x \in E^*_{\gamma\lambda} \cap Q_j : \mathcal{G}_{\mathrm{P},2,j,N} f(x) > \lambda\}| \le \frac{1}{\lambda^2} \int_{E^*_{\gamma\lambda}\cap Q_j} \mathcal{G}_{\mathrm{P},2,j,N} f(x)^2 dx \\
&\qquad \le \frac{1}{\lambda^2} \int_{E^*_{\gamma\lambda}\cap Q_j} \int_0^{\frac{\ell(Q_j)}{2}} \int_{B(x,t)} \left| t\nabla_{y,t} u(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}}\, dx =: \frac{1}{\lambda^2} \int_{E^*_{\gamma\lambda}\cap Q_j} \mathcal{G}_{\mathrm{P},2} f(x)^2\, dx. \quad (3.33)
\end{aligned}
$$

In order to estimate the last integral above, for $0 < \varepsilon < \frac{\ell(Q_j)}{2}$, consider the function

$$
\mathcal{G}_{\mathrm{P},2,\varepsilon} f(x) := \left(\int_\varepsilon^{\frac{\ell(Q_j)}{2}} \int_{B(x,t)} \left| t\nabla_{y,t} u(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}. \tag{3.34}
$$

Besides, for $\beta > 0$, consider the region

$$
\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda} \cap Q_j) := \bigcup_{x\in E^*_{\gamma\lambda}\cap Q_j} \left\{ (y,t) \in \mathbb{R}^n \times (\beta\varepsilon, \beta\ell(Q_j)) : |y-x| < \frac{t}{\beta} \right\},
$$

and set

$$
B(y) := \begin{pmatrix} A(y) & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.35}
$$

where A is as in (1.19). Then, we have that there exist $0 < \widetilde{\lambda} \le \widetilde{\Lambda} < \infty$ such that

$$
\widetilde{\lambda}\, |\xi|^2 \le \operatorname{Re} B(x)\, \xi \cdot \bar{\xi} \tag{3.36}
$$

and

$$
|B(x)\, \xi \cdot \bar{\zeta}| \le \widetilde{\Lambda}\, |\xi|\, |\zeta|, \tag{3.37}
$$

for all $\xi, \zeta \in \mathbb{C}^{n+1}$ and almost every $x \in \mathbb{R}^n$. Moreover, we have that

$$
\partial_t u(y,t) = \operatorname*{div}_{y,t}(tB(y)\, \nabla_{y,t} u(y,t)). \tag{3.38}
$$

Finally notice that

$$
\mathcal{G}_{P,2,\varepsilon} f(x)^2 \lesssim \int_{\beta\varepsilon}^{\beta\ell(Q_j)} \int_{|x-y|<\frac{t}{\beta}} |t\nabla_{y,t} u(y,t)|^2 \frac{dy\,dt}{t^{n+1}}, \quad \text{for all} \quad \beta \in (2^{-1}, 1).
$$

From this we immediately see

$$
\int_{E^*_{\gamma\lambda}\cap Q_j} \mathcal{G}_{P,2,\varepsilon} f(x)^2 dx \lesssim \iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_{y,t} u(y,t)|^2 dy\,dt, \quad \text{for all} \quad \beta \in (2^{-1}, 1). \tag{3.39}
$$

To bound the last integral above apply (3.36) and integration by parts:

$$
\begin{aligned}
&\iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_{y,t} u(y,t)|^2 dy\,dt \lesssim \operatorname{Re} \iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} tB(y)\nabla_{y,t} u(y,t) \cdot \overline{\nabla_{y,t} u(y,t)} dy\,dt \\
&= \frac{1}{2} \iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} \left[tB(y)\nabla_{y,t} u(y,t) \cdot \overline{\nabla_{y,t} u(y,t)} + t\overline{B(y)\nabla_{y,t} u(y,t)} \cdot \nabla_{y,t} u(y,t)\right] dy\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= C \iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} \Big[-\operatorname*{div}_{y,t}(tB(y)\nabla_{y,t}u(y,t))\overline{u(y,t)} - \overline{\operatorname*{div}_{y,t}(tB(y)\nabla_{y,t}u(y,t))}u(y,t)\Big]\,dy\,dt \\
&\quad + \int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} \Big[tB(y)\nabla_{y,t}u(y,t)\cdot \upsilon_{y,t}(y,t)\overline{u(y,t)} + t\overline{B(y)\nabla_{y,t}u(y,t)}\cdot\upsilon_{y,t}(y,t)u(y,t)\Big]\,d\sigma,
\end{aligned}
$$

where $\upsilon_{y,t}$ is the outer unit normal associated with the domain of integration.

Now, use (3.38) in the first integral, (3.37) in the second one, and the fact that $|\upsilon_{y,t}(y,t)| = 1$, to obtain

$$
\begin{aligned}
\iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_{y,t}u(y,t)|^2 dy\,dt &\lesssim \left|\iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} \Big[-\partial_t u(y,t)\cdot\overline{u(y,t)} - \partial_t\overline{u(y,t)}\cdot u(y,t)\Big]\,dy\,dt\right| \\
&\quad + \int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_{y,t}u(y,t)||u(y,t)|\,d\sigma \\
&= \left| -\iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} \partial_t|u(y,t)|^2 dy\,dt\right| + \int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_{y,t}u(y,t)||u(y,t)|\,d\sigma.
\end{aligned}
$$

Then, applying again integration by parts and Cauchy-Schwarz's inequality, we conclude that

$$
\begin{aligned}
&\iint_{\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_{y,t}u(y,t)|^2 dy\,dt \\
&\qquad \le \int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} |u(y,t)|^2 d\sigma + \int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_{y,t}u(y,t)||u(y,t)|\,d\sigma \\
&\qquad \lesssim \int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} |u(y,t)|^2 d\sigma + \int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} |t\nabla_{y,t}u(y,t)|^2 d\sigma.
\end{aligned}
\tag{3.40}
$$

Now, observe that

$$
\begin{aligned}
\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j) &= \left\{(y,t)\in\mathbb{R}^{n+1}_+ : d(y,Q_j\cap E^*_{\gamma\lambda}) = \frac{t}{\beta},\ \beta\varepsilon\le t\le\beta\ell(Q_j)\right\} \\
&\quad \bigcup \left\{y\in\mathbb{R}^n : d(y,Q_j\cap E^*_{\gamma\lambda})<\varepsilon\right\}\times\{\beta\varepsilon\} \\
&\quad \bigcup \left\{y\in\mathbb{R}^n : d(y,Q_j\cap E^*_{\gamma\lambda})<\ell(Q_j)\right\}\times\left\{\beta\ell(Q_j)\right\} \\
&=: \mathcal{H}(\beta)\cup\mathcal{T}(\varepsilon)\times\{\beta\varepsilon\}\cup\mathcal{T}(\ell(Q_j))\times\left\{\beta\ell(Q_j)\right\},
\end{aligned}
$$

and for every function $h : \mathbb{R}^{n+1}_+ \to \mathbb{R}$

$$
\int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} h\,d\sigma = \int_{\mathcal{H}(\beta)} h\,d\sigma + \int_{\mathcal{T}(\varepsilon)} h(y,\beta\varepsilon)\,dy + \int_{\mathcal{T}(\ell(Q_j))} h(y,\beta\ell(Q_j))\,dy.
$$

Besides, consider

$$
\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j) := \left\{(y,t)\in\mathbb{R}^n\times(2^{-1}\varepsilon,\ell(Q_j)) : 2^{-1}d(y,E^*_{\gamma\lambda}\cap Q_j) < t < d(y,E^*_{\gamma\lambda}\cap Q_j)\right\}
$$

and $F(y,t) := \frac{d(y,Q_j\cap E^*_{\gamma\lambda})}{t}$. We have that

$$
|JF(y,t)| \le \frac{1}{|t|} + \frac{d(y,Q_j\cap E^*_{\gamma\lambda})}{t^2}, \quad t\neq 0, \ \text{ for a.e. } y\in\mathbb{R}^n,
$$

where JF denotes the Jacobian of F. Then, integrating in $\beta\in(1/2,1)$ and applying the coarea formula

$$
\int_{\frac{1}{2}}^1 \int_{\mathcal{H}(\beta)} h\,d\sigma\,d\beta \le \int_{\frac{1}{2}}^1 \int_{F^{-1}(1/\beta)} h\mathbf{1}_{\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)}\,d\sigma\,d\beta
$$

$$
\begin{aligned}
&\leq \iint_{\mathbb{R}^{n+1}_+} h(y,t)\mathbf{1}_{\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)}(y,t)|JF(y,t)|dy\,dt \\
&\leq \iint_{\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} h(y,t)|JF(y,t)|dy\,dt \\
&\leq \iint_{\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} h(y,t)\frac{1}{t}\left(1+\frac{d(y,Q_j\cap E^*_{\gamma\lambda})}{t}\right)dy\,dt \\
&\lesssim \iint_{\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} h(y,t)\frac{dy\,dt}{t}.
\end{aligned}
$$

On the other hand, doing the change of variables $\beta\varepsilon = t$, we have

$$
\int_{\frac{1}{2}}^{1}\int_{\mathcal{T}(\varepsilon)} h(y,\beta\varepsilon)dy\,d\beta = \int_{\frac{\varepsilon}{2}}^{\varepsilon}\frac{1}{\varepsilon}\int_{\mathcal{T}(\varepsilon)} h(y,t)dy\,dt \lesssim \iint_{\mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)} h(y,t)\frac{dy\,dt}{t},
$$

where

$$
\mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j) := \left\{(y,t)\in\mathbb{R}^n\times(2^{-1}\varepsilon,\varepsilon) : d(y,E^*_{\gamma\lambda}\cap Q_j) < 2t\right\},
$$

And finally, doing the change of variables $\beta\ell(Q_j) = t$ we obtain

$$
\int_{\frac{1}{2}}^{1}\int_{\mathcal{T}(\ell(Q_j))} h(y,\beta\ell(Q_j))dy\,d\beta = \int_{\frac{\ell(Q_j)}{2}}^{\ell(Q_j)}\frac{1}{\ell(Q_j)}\int_{\mathcal{T}(\ell(Q_j))} h(y,t)dy\,dt \lesssim \iint_{\mathcal{B}^{\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} h(y,t)\frac{dy\,dt}{t},
$$

where

$$
\mathcal{B}^{\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j) := \left\{(y,t)\in\mathbb{R}^n\times(2^{-1}\ell(Q_j),\ell(Q_j)) : d(y,E^*_{\gamma\lambda}\cap Q_j) < 2t\right\}.
$$

Therefore, applying the previous estimates with $h(y,t) = |u(y,t)|^2$, and $h(y,t) = |t\nabla_{y,t}u(y,t)|^2$, and also (3.39) and (3.40), we have

$$
\begin{aligned}
\int_{E^*_{\gamma\lambda}\cap Q_j} \mathcal{G}_{P,2,\varepsilon}f(x)^2dx &= 2\int_{\frac{1}{2}}^{1}\int_{E^*_{\gamma\lambda}\cap Q_j} \mathcal{G}_{P,2,\varepsilon}f(x)^2dx\,d\beta \\
&\lesssim \int_{\frac{1}{2}}^{1}\int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} |u(y,t)|^2d\sigma\,d\beta + \int_{\frac{1}{2}}^{1}\int_{\partial\mathcal{R}^{\varepsilon,\ell(Q_j),\beta}(E^*_{\gamma\lambda}\cap Q_j)} |t\nabla_{y,t}u(y,t)|^2d\sigma\,d\beta \\
&\lesssim \iint_{\widetilde{\mathcal{B}}(E^*_{\gamma\lambda}\cap Q_j)} |u(y,t)|^2\frac{dy\,dt}{t} + \iint_{\widetilde{\mathcal{B}}(E^*_{\gamma\lambda}\cap Q_j)} |t\nabla_{y,t}u(y,t)|^2\frac{dy\,dt}{t} \\
&=: I + II, \qquad (3.41)
\end{aligned}
$$

where

$$
\widetilde{\mathcal{B}}(E^*_{\gamma\lambda}\cap Q_j) := \mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)\cup\mathcal{B}^{\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)\cup\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j).
$$

Hence,

$$
\begin{aligned}
I \lesssim \iint_{\mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)} |u(y,t)|^2\frac{dy\,dt}{t} &+ \iint_{\mathcal{B}^{\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} |u(y,t)|^2\frac{dy\,dt}{t} \\
&+ \iint_{\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} |u(y,t)|^2\frac{dy\,dt}{t} =: I_1 + I_2 + I_3,
\end{aligned}
$$

and

$$II \lesssim \iint_{\mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)} |t\nabla_{y,t}u(y,t)|^2 \frac{dy\,dt}{t} + \iint_{\mathcal{B}^{\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} |t\nabla_{y,t}u(y,t)|^2 \frac{dy\,dt}{t}$$
$$+ \iint_{\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j)} |t\nabla_{y,t}u(y,t)|^2 \frac{dy\,dt}{t} =: II_1 + II_2 + II_3.$$

We start estimating I_1. For every $(y,t) \in \mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)$, there exists $y_0 \in E^*_{\gamma\lambda}\cap Q_j$ such that $y \in B(y_0, 2t)$. Besides, since $y_0 \in E^*_{\gamma\lambda}\cap Q_j$, from the definition of $E^*_{\gamma\lambda}$ we have that $|E_{\gamma\lambda}\cap B(y_0,2t)| \geq Ct^n$ and then $|E_{\gamma\lambda}\cap B(y,4t)| \geq Ct^n$. Thus, we have for $\kappa \geq 4$,

$$I_1 \lesssim \iint_{\mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)} \int_{E_{\gamma\lambda}\cap B(y,4t)} |u(y,t)|^2 dx \frac{dy\,dt}{t^{n+1}} \lesssim \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_{8Q_j\cap E_{\gamma\lambda}} \int_{B(x,4t)} |u(y,t)|^2 \frac{dy}{t^n}\frac{dx\,dt}{t}$$
$$\leq \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_{8Q_j\cap E_{\gamma\lambda}} \mathcal{N}^{\kappa}_{\mathrm{P}} f(x)^2 \frac{dx\,dt}{t} \lesssim |Q_j|(\gamma\lambda)^2.$$

The second inequality follows applying Fubini and noticing that $(y,t) \in \mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)$ and $x \in E_{\gamma\lambda}\cap B(y,4t)$ imply that $x \in E_{\gamma\lambda}\cap 8Q_j$, $y \in B(x,4t)$, and $t \in \left(\frac{\varepsilon}{2},\varepsilon\right)$, where we recall that $\varepsilon < \frac{\ell(Q_j)}{2}$. Similarly, for II_1,

$$II_1 \lesssim \iint_{\mathcal{B}^{\varepsilon}(E^*_{\gamma\lambda}\cap Q_j)} \int_{E_{\gamma\lambda}\cap B(y,4t)} |t\nabla_{y,t}u(y,t)|^2 dx \frac{dy\,dt}{t^{n+1}} \lesssim \int_{8Q_j\cap E_{\gamma\lambda}} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_{B(x,4t)} |\nabla_{y,t}u(y,t)|^2 \frac{dy\,dt}{t^{n-1}} dx.$$

Now, consider the elliptic operator $\widetilde{L}u(y,t) := -\operatorname{div}_{y,t}\left(B(y)\nabla_{y,t}u(y,t)\right)$, (where B is the matrix defined in (3.35)). Besides, for each $x \in 8Q_j\cap E_{\gamma\lambda}$, cover the truncated cone $\Gamma^{\frac{\varepsilon}{2},\varepsilon,4}(x) := \{(y,t) \in \mathbb{R}^n \times (\varepsilon/2,\varepsilon) : |x-y| < 4t\}$ by dyadic cubes $R_i \subset \mathbb{R}^{n+1}_+$, of side length ℓ_ε, $\frac{\varepsilon}{16\sqrt{n}} < \ell_\varepsilon \leq \frac{\varepsilon}{8\sqrt{n}}$. Then, the family $\{2R_i\}_{i\in\mathbb{N}}$ has controlled overlap. Hence since $\widetilde{L}u = 0$, we can apply Caccioppoli's inequality and obtain for $\kappa \geq 5$

$$\int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_{B(x,4t)} |\nabla_{y,t}u(y,t)|^2 \frac{dy\,dt}{t^{n-1}} \lesssim \frac{1}{\varepsilon^{n-1}} \sum_{i=1}^{M} \iint_{R_i} |\nabla_{y,t}u(y,t)|^2 dy\,dt$$
$$\lesssim \frac{1}{\varepsilon^{n+1}} \sum_{i=1}^{M} \iint_{2R_i} |u(y,t)|^2 dy\,dt \lesssim \frac{1}{\varepsilon^{n+1}} \int_{\frac{\varepsilon}{4}}^{2\varepsilon} \int_{B(x,5t)} |u(y,t)|^2 dy\,dt$$
$$\lesssim \frac{1}{\varepsilon^{n+1}} \int_{\frac{\varepsilon}{4}}^{2\varepsilon} t^n\,dt\, \mathcal{N}^{\kappa}_{\mathrm{P}} f(x)^2 \lesssim \mathcal{N}^{\kappa}_{\mathrm{P}} f(x)^2.$$

Consequently,

$$II_1 \lesssim \int_{8Q_j\cap E_{\gamma\lambda}} \mathcal{N}^{\kappa}_{\mathrm{P}} f(x)^2\,dx \lesssim |Q_j|(\gamma\lambda)^2.$$

Arguing in the same way, we obtain for I_2 and II_2 similar estimates:

$$I_2 \lesssim |Q_j|(\gamma\lambda)^2 \quad \text{and} \quad II_2 \lesssim |Q_j|(\gamma\lambda)^2.$$

Finally, for I_3 and II_3, decompose $\mathbb{R}^n \setminus (E^*_{\gamma\lambda}\cap Q_j) = O^*_{\gamma\lambda} \cup (\mathbb{R}^n \setminus Q_j)$, which is an open set since the cubes Q_j are closed, into a family of Whitney balls $\{B(x_k,r_k)\}_{k=0}^{\infty}$, such that $\bigcup_{k=0}^{\infty} B(x_k,r_k) = O^*_{\gamma\lambda} \cup (\mathbb{R}^n \setminus Q_j)$, and for some constants $0 < c_1 < c_2 < 1$ and $c_3 \in \mathbb{N}$, $c_1 d(x_k, E^*_{\gamma\lambda}\cap Q_j) \leq r_k \leq c_2 d(x_k, E^*_{\gamma\lambda}\cap Q_j)$, and $\sum_{k=0}^{\infty} \mathbf{1}_{B(x_k,r_k)}(x) \leq c_3$, for all $x \in \mathbb{R}^n$. Besides, consider the set

$$\widetilde{K} := \{k : d(x_k, E^*_{\gamma\lambda}\cap Q_j)) \leq 2(1-c_2)^{-1}\ell(Q_j)\}.$$

We are going to see that

$$\mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda}\cap Q_j) \subset \bigcup_{k\in\widetilde{K}} B(x_k,r_k) \times [r_k(c_2^{-1}-1)/2, r_k(c_1^{-1}+1)]. \tag{3.42}$$

Indeed, for $(y,t) \in \mathcal{B}^{\varepsilon,\ell(Q_j)}(E^*_{\gamma\lambda} \cap Q_j)$, we have that $\varepsilon/2 < t < \ell(Q_j)$, $y \in \mathbb{R}^n \setminus (E^*_{\gamma\lambda} \cap Q_j)$, and

$$2^{-1} d(y, E^*_{\gamma\lambda} \cap Q_j) < t < d(y, E^*_{\gamma\lambda} \cap Q_j). \tag{3.43}$$

Then, there exists k such that $y \in B(x_k, r_k)$. We see that $k \in \widetilde{K}$ and $r_k(c_2^{-1} - 1)/2 \le t \le r_k(c_1^{-1} + 1)$. On the one hand, we have

$$d(y, E^*_{\gamma\lambda} \cap Q_j) \le |y - x_k| + d(x_k, E^*_{\gamma\lambda} \cap Q_j) \le r_k + c_1^{-1} r_k = (1 + c_1^{-1}) r_k,$$

and, on the other hand,

$$d(y, E^*_{\gamma\lambda} \cap Q_j) \ge d(x_k, E^*_{\gamma\lambda} \cap Q_j) - |y - x_k| \ge (r_k c_2^{-1} - r_k) = (c_2^{-1} - 1) r_k.$$

Therefore, by (3.43), we have that $t \in [r_k(c_2^{-1} - 1)/2, r_k(c_1^{-1} + 1)]$. From this and recalling that $t < \ell(Q_j)$, we have

$$d(x_k, E^*_{\gamma\lambda} \cap Q_j) \le |y - x_k| + d(y, E^*_{\gamma\lambda} \cap Q_j) \le r_k + 2\ell(Q_j) \le \frac{2t}{(c_2^{-1} - 1)} + 2\ell(Q_j) \le 2(1 - c_2)^{-1} \ell(Q_j),$$

which in turn gives us that $k \in \widetilde{K}$. Moreover, note that for every $k \in \widetilde{K}$, we have that

$$B(x_k, r_k) \subset C(c_2) Q_j, \quad \text{with} \quad C(c_2) := 4(1 - c_2)^{-1}(c_2 + 1) + 1.$$

Indeed, note that since $E^*_{\gamma\lambda} \cap Q_j \subset Q_j$ then $d(x_k, Q_j) \le d(x_k, E^*_{\gamma\lambda} \cap Q_j)$. Hence, for $x_0 \in B(x_k, r_k)$ and x_{Q_j} being the center of Q_j, we have,

$$\begin{aligned}
|x_0 - x_{Q_j}|_\infty \le |x_0 - x_k|_\infty + |x_k - x_{Q_j}|_\infty &\le r_k + \left(2(1 - c_2)^{-1} + \frac{1}{2}\right) \ell(Q_j) \\
&\le \left(2(1 - c_2)^{-1}(c_2 + 1) + \frac{1}{2}\right) \ell(Q_j) = (4(1 - c_2)^{-1}(c_2 + 1) + 1) \frac{\ell(Q_j)}{2}.
\end{aligned}$$

Now, since $E^*_{\gamma\lambda} \subset E_{\gamma\lambda}$ then

$$d(x_k, Q_j \cap E_{\gamma\lambda}) \le d(x_k, E^*_{\gamma\lambda} \cap Q_j) \le c_1^{-1} r_k \le \frac{2c_2}{c_1(1 - c_2)} t,$$

which implies that, for $\kappa > \frac{2c_2}{c_1(1-c_2)}$ there exists $\widetilde{x} \in Q_j \cap E_{\gamma\lambda}$ such that $|\widetilde{x} - x_k| < \kappa t$, and then

$$\int_{B(x_k, \frac{2c_2}{1-c_2} t)} |u(y,t)|^2 \frac{dy}{t^n} \le \int_{B(x_k, \frac{2c_2}{c_1(1-c_2)} t)} |u(y,t)|^2 \frac{dy}{t^n} \le \int_{B(x_k, \kappa t)} |u(y,t)|^2 \frac{dy}{t^n} \le \mathcal{N}^\kappa_{\mathrm{P}} f(\widetilde{x}) \le (\gamma\lambda)^2.$$

Therefore, we have

$$\begin{aligned}
I_3 &\le \sum_{k \in \widetilde{K}} \int_{\frac{r_k(c_2^{-1}-1)}{2}}^{r_k(c_1^{-1}+1)} \int_{B(x_k, r_k)} |u(y,t)|^2 \frac{dy\,dt}{t} \lesssim \sum_{k \in \widetilde{K}} r_k^n \int_{\frac{r_k(c_2^{-1}-1)}{2}}^{r_k(c_1^{-1}+1)} \int_{B(x_k, \frac{2c_2}{1-c_2} t)} |u(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \\
&\lesssim (\gamma\lambda)^2 \sum_{k \in \widetilde{K}} r_k^n \lesssim (\gamma\lambda)^2 \left| \bigcup_{k \in \widetilde{K}} B(x_k, r_k) \right| \lesssim |Q_j| (\gamma\lambda)^2.
\end{aligned}$$

Similarly, for II_3, we obtain

$$II_3 \le \sum_{k \in \widetilde{K}} \int_{\frac{r_k(c_2^{-1}-1)}{2}}^{r_k(c_1^{-1}+1)} \int_{B(x_k, r_k)} |t \nabla_{y,t} u(y,t)|^2 \frac{dy\,dt}{t} \lesssim \sum_{k \in \widetilde{K}} r_k^n \int_{\frac{r_k(c_2^{-1}-1)}{2}}^{r_k(c_1^{-1}+1)} \int_{B(x_k, \frac{2c_2}{1-c_2} t)} |t \nabla_{y,t} u(y,t)|^2 \frac{dy\,dt}{t^{n+1}}.$$

Then, arguing as in the estimate of II_1 (taking κ larger if necessary), we conclude that

$$II_3 \lesssim |Q_j|(\gamma\lambda)^2.$$

Gathering the estimates obtained for I and II give us

$$\int_{E^*_{\gamma\lambda}\cap Q_j} \mathcal{G}_{\mathrm{P},2,\varepsilon} f(x)^2 dx \leq C|Q_j|(\gamma\lambda)^2,$$

with C independent of ε. Now, recall the definitions of $\mathcal{G}_{\mathrm{P},2}$ and $\mathcal{G}_{\mathrm{P},2,\varepsilon}$ in (3.33) and (3.34) respectively. Then, let $\varepsilon \to 0$ and obtain

$$\int_{E^*_{\gamma\lambda}\cap Q_j} \mathcal{G}_{\mathrm{P},2} f(x)^2 dx \leq C|Q_j|(\gamma\lambda)^2.$$

This, together with (3.33), yields (3.32).

In order to complete the proof of Theorem 3.29, we need to establish part (b). The argument follows the lines of [55, Theorem 6.1] and the proof of part (a), so we only sketch the main changes. Consider, for each $N > 1$, K_N as in (3.30), $\alpha \geq 1$, and

$$\mathrm{G}^\alpha_{\mathrm{H},N} f(x) := \left(\iint_{\Gamma^\alpha(x)} \mathbf{1}_{K_N}(y,t) \left| t\nabla_y e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

We write $\mathrm{G}_{\mathrm{H},N}$ when $\alpha = 1$. Notice that $\operatorname{supp} \mathrm{G}_{\mathrm{H},N} f \subset B(0,(\alpha+1)N)$ and much as before

$$\|\mathrm{G}^\alpha_{\mathrm{H},N} f\|_{L^p(w)} \leq C\|f\|_{L^2(\mathbb{R}^n)} N^{\frac{n}{2}} w(B(0,(\alpha+1)N))^{\frac{1}{p}} < \infty.$$

Hence, it is enough to show part (b) with $\mathrm{G}_{\mathrm{H},N}$ in place of G_{H} with constants uniform in N.

We follow the proof of part (a), replacing $\mathcal{G}^\alpha_{\mathrm{P},N}$ and $\mathcal{N}_{\mathrm{P}}$ with $\mathrm{G}^\alpha_{\mathrm{H},N}$ and $\mathcal{N}_{\mathrm{H}}$, respectively ($\mathcal{G}_{\mathrm{P},N}$ with $\mathrm{G}_{\mathrm{H},N}$ when $\alpha = 1$). We also need to replace $u(y,t)$ with $v(y,t) := e^{-t^2 L} f(y)$ and $t\nabla_{y,t} u(y,t)$ with $t\nabla_y v(y,t)$.

We also use the ellipticity of the matrix A (see (1.18)) instead of the properties of the block matrix B defined in (3.35). Then, we have that

$$\int_{E^*_{\gamma\lambda}\cap Q_j} \mathrm{G}_{\mathrm{H},2,\varepsilon} f(x)^2 dx \lesssim \iint_{\widetilde{\mathcal{B}}(E^*_{\gamma\lambda}\cap Q_j)} |v(y,t)|^2\, dy\,dt + \iint_{\widetilde{\mathcal{B}}(E^*_{\gamma\lambda}\cap Q_j)} t|\nabla_y v(y,t)|^2\, dy\,dt =: \widetilde{I} + \widetilde{II}.$$

From here the proof proceeds much as the proof of part (a): term $\widetilde{I}$ is estimated as term I, and term $\widetilde{II}$ as term II but, in this case, as in the proof of [55, Theorem 6.1], we need to use the following parabolic Caccioppoli inequality (see [55, Lemma 2.8]) formulated in the next lemma.

Lemma 3.44. *Suppose $\partial_t f = -Lf$ in $I_{2r}(x_0,t_0)$, where $I_r(x_0,t_0) = B(x_0,r) \times [t_0 - cr^2, t_0]$, $t_0 > 4cr^2$ and $c > 0$. Then, there exists $C = C(\lambda,\Lambda,c) > 0$ such that*

$$\iint_{I_r(x_0,t_0)} |\nabla_x f(x,t)|^2 dx\,dt \leq \frac{C}{r^2} \iint_{I_{2r}(x_0,t_0)} |f(x,t)|^2 dx\,dt.$$

□

Remark 3.45. *Following the explanation of Remark 3.22, one can see that Theorem 3.29 holds for all functions $f \in L^q(w)$ with $w \in A_\infty$ and $q \in \mathcal{W}_w(p_-(L), p_+(L))$. Details are left to the interested reader.*

3.3 Some further remarks

Proposition 3.46. *Given $w \in A_\infty$, such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, then*

$$\mathcal{W}_w(p_-(L), p_+(L)) \subseteq \{p : \|\mathcal{T} f\|_{L^p(w)} \approx \|f\|_{L^p(w)}, \ \forall f \in L^p(w)\}, \tag{3.47}$$

where $\mathcal{T}$ is any of the square functions considered in (1.26)-(1.31) *or a non-tangential maximal function defined in* (1.32).

Proof. Take $p \in \mathcal{W}_w(p_-(L), p_+(L))$ and $f \in L^p(w)$. From Theorems 3.1, 3.2, and 3.25, and Remarks 3.22 and 3.45, we have that

$$\|\mathcal{T} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

In order to show the converse inequality, let us first consider the particular case of $\mathcal{T} = \mathcal{S}_{m,\mathrm{P}}$, for $m \in \mathbb{N}$.

Note that we have the following Calderón reproducing formula for f,

$$f(x) = C_m \int_0^\infty \left((t^2L)^m e^{-t\sqrt{L}}\right)^2 f(x)\frac{dt}{t} = C_m \lim_{N\to\infty} \int_{N^{-1}}^{N} \left((t^2L)^m e^{-t\sqrt{L}}\right)^2 f(x)\frac{dt}{t}, \tag{3.48}$$

where the equalities are in $L^p(w)$.

Remark 3.49. *A priori, by $L^2(\mathbb{R}^n)$ functional calculus, we have the above equalities for functions in $L^2(\mathbb{R}^n)$. Here we explain how to extend them to functions in $L^p(w)$ for all $p \in \mathcal{W}_w(p_-(L), p_+(L))$, where, abusing notation, we understand that L is the infinitesimal generator on $L^p(w)$ of the heat semigroup $\{e^{-tL}\}_{t>0}$ (see [11, Remark 3.5]). Fixing such a p, we first introduce the operator $\mathcal{T}_{t,L} := (t^2L)^m e^{-t\sqrt{L}}$, whose adjoint (in $L^2(\mathbb{R}^n)$) is $\mathcal{T}_{t,L}^* = (t^2L^*)^m e^{-t\sqrt{L^*}} =: \mathcal{T}_{t,L^*}$. Next, we set $Q_L f(x,t) := \mathcal{T}_{t,L^*} f(x)$ for $(x,t) \in \mathbb{R}^{n+1}_+$ and $f \in L^2(\mathbb{R}^n)$. Since $p \in \mathcal{W}_w(p_-(L), p_+(L))$ then $p' \in \mathcal{W}_{w^{1-p'}}(p_-(L^*), p_+(L^*))$, by [9, Lemma 4.4] and the fact that $p_\pm(L^*) = p_\mp(L)'$ (see [3]), thus, the vertical square function defined by $\mathcal{T}_{t,L^*}$ is bounded on $L^{p'}(w^{1-p'})$ (see* (4.57) *and [11]). Additionally, writing $\mathbb{H} := L^2\left((0,\infty), \frac{dt}{t}\right)$, we obtain*

$$\|Q_L h\|_{L^{p'}_{\mathbb{H}}(w^{1-p'})} = \big\| \|Q_L h\|_{\mathbb{H}} \big\|_{L^{p'}(w^{1-p'})} = \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |\mathcal{T}_{t,L^*} h(x)|^2 \frac{dt}{t}\right)^{\frac{p'}{2}} w(x)^{1-p'} dx\right)^{\frac{1}{p'}} \lesssim \|h\|_{L^{p'}(w^{1-p'})}. \tag{3.50}$$

Therefore, $Q_L : L^{p'}(w^{1-p'}) \to L^{p'}_{\mathbb{H}}(w^{1-p'})$. Besides, if Q_L^* is its adjoint operator with respect to dx, for $h \in L^2_{\mathbb{H}}(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$, we have that

$$\langle Q_L^* h, f\rangle_{L^2(\mathbb{R}^n)} = \langle h, Q_L f\rangle_{L^2_{\mathbb{H}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_0^\infty h(y,t)\overline{(\mathcal{T}_{t,L})^* f(y)}\frac{dt}{t}dy = \int_{\mathbb{R}^n}\int_0^\infty \mathcal{T}_{t,L} h(y,t)\frac{dt}{t}\overline{f(y)}dy,$$

where it is implicitly understood that $\mathcal{T}_{t,L} h(y,t) = \mathcal{T}_{t,L}\big(h(\cdot,t)\big)(y)$. Consequently, for every $h \in L^2_{\mathbb{H}}(\mathbb{R}^n)$,

$$Q_L^* h(x) = \int_0^\infty \mathcal{T}_{t,L} h(x,t)\frac{dt}{t} = \int_0^\infty (t^2L)^m e^{-t\sqrt{L}} h(x,t)\frac{dt}{t},$$

and

$$\left|\int_{\mathbb{R}^n} Q_L^* h(x) f(x) dx\right| \leq \int_{\mathbb{R}^n} \|h\|_{\mathbb{H}} \|Q_L^M f\|_{\mathbb{H}} dy.$$

Hence, by the boundedness of Q_L and by a density argument, we conclude that Q_L^* has a bounded extension from $L^p_{\mathbb{H}}(w)$ to $L^p(w)$ (see also [3, 11]).

Moreover, note that $C_m Q_L^* Q_{L^*} f = f$ for every $f \in L^2(\mathbb{R}^n)$, where according to the notation introduced above $Q_{L^*} f(x,t) = \mathcal{T}_{t,L} f(x) = (t^2L)^m e^{-t\sqrt{L}} f(x)$. On the other hand, for every $f \in L^p(w)$ and $g \in L^2(\mathbb{R}^n) \cap L^p(w)$ we have that

$$\begin{aligned}\left\|f - C_m Q_L^* Q_{L^*} f\right\|_{L^p(w)} &\le \|f-g\|_{L^p(w)} + C_m \left\|Q_L^* Q_{L^*}(g-f)\right\|_{L^p(w)} \\ &\lesssim \|f-g\|_{L^p(w)} + \|Q_{L^*}(g-f)\|_{L^p_{\mathbb{H}}(w)} \lesssim \|f-g\|_{L^p(w)},\end{aligned}$$

where we have used the boundedness of Q_L^* along with the fact that Q_{L^*} is bounded from $L^p(w)$ to $L^p_{\mathbb{H}}(w)$, the latter follows as in (3.50) *with $L^p(w)$ in place of $L^{p'}(w^{1-p'})$ since $p \in \mathcal{W}_w(p_-(L), p_+(L))$. Using now that $L^2(\mathbb{R}^n) \cap L^p(w)$ is dense in $L^p(w)$ we easily conclude the first equality in* (3.48).

In order to obtain the second equality in (3.48) *we write $I_N := [N^{-1}, N]$ and observe that for every $h \in L^p_{\mathbb{H}}(w)$, one has that $\mathbf{1}_{I_N} h \longrightarrow h$ in $L^p_{\mathbb{H}}(w)$ as $N \to \infty$, and therefore $Q_L^*(\mathbf{1}_{I_N} h) \longrightarrow Q_L^* h$ in $L^p(w)$ as $N \to \infty$. Taking now $f \in L^p(w)$, as mentioned above, $Q_{L^*} f \in L^p_{\mathbb{H}}(w)$ and it follows that $Q_L^*(\mathbf{1}_{I_N} Q_{L^*} f) \longrightarrow Q_L^*(Q_{L^*} f)$ on $L^p(w)$, from this we obtain the second equality in* (3.48).

Moreover, note that the operator Q_L defined in the above remark, is also bounded from $L^{p'}(w^{1-p'})$ to $T^{p'}(w^{1-p'})$, for all $p' \in \mathcal{W}_{w^{1-p'}}(p_-(L^*), p_+(L^*))$. Indeed, by Theorem 3.2 (and the observations made on Remark 3.49) we have that, for $h \in L^{p'}(w^{1-p'})$,

$$\|Q_L h\|_{T^{p'}(w^{1-p'})} = \left(\int_{\mathbb{R}^n} \left(\iint_{\Gamma(x)} |(t^2 L^*)^m e^{-t\sqrt{L^*}} h(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{p'}{2}} w^{1-p'}(x) dx\right)^{\frac{1}{p'}} \lesssim \|h\|_{L^{p'}(w^{1-p'})}.$$

Again as in Remark 3.49, we have for all $H \in T^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$,

$$Q_L^* H(x) = \int_0^\infty (t^2 L)^m e^{-t\sqrt{L}} H(x,t) \frac{dt}{t},$$

and

$$\left|\int_{\mathbb{R}^n} Q_L^* H(y) g(y) dy\right| = \left|\int_{\mathbb{R}^n} \int_0^\infty H(y,t) (t^2 L^*)^m e^{-t\sqrt{L^*}} \bar{g}(y) \int_{B(y,t)} dx \frac{dt\,dy}{t^{n+1}}\right| \le \int_{\mathbb{R}^n} |||H|||_{\Gamma(x)} |||\mathcal{T}_t^* \bar{g}|||_{\Gamma(x)} dx,$$

where recall that Q_L^* is the adjoint of Q_l, and the definition of $|||\cdot|||_{\Gamma(x)}$ in Notation (h). Hence, by the boundedness of Q_L from $L^{p'}(w^{1-p'})$ to $T^{p'}(w^{1-p'})$, for all $p' \in \mathcal{W}_{w^{1-p'}}(p_-(L^*), p_+(L^*))$ and by a density argument, we conclude that Q_L^* has a bounded extension from $T^p(w)$ to $L^p(w)$, for all $p \in \mathcal{W}_w(p_-(L), p_+(L))$.

Consequently, for $g \in L^{p'}(\mathbb{R}^n)$, $f \in L^p(w)$ and $\mathcal{T}_{t,L} f = (t^2 L)^m e^{-t\sqrt{L}} f \in T^p(w)$ (by Theorem 3.2), we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x) \bar{g}(x) w(x)^{\frac{1}{p}}\, dx\right| &= C_m \left|\int_{\mathbb{R}^n} Q_L^* F(x) \bar{g}(x) w(x)^{\frac{1}{p}}\, dx\right| \\ &= C_m \left|\int_{\mathbb{R}^n} \int_0^\infty (t^2 L)^m e^{-t\sqrt{L}} F(x,t) \frac{dt}{t} \bar{g}(x) w(x)^{\frac{1}{p}}\, dx\right| \\ &= C_m \left|\int_{\mathbb{R}^n} \int_0^\infty F(x,t) \overline{(t^2 L^*)^m e^{-t\sqrt{L^*}} (g w^{\frac{1}{p}})(x)} \frac{dt}{t}\, dx\right| \\ &= C_m \left|\int_{\mathbb{R}^n} \int_0^\infty F(x,t) \overline{(t^2 L^*)^m e^{-t\sqrt{L^*}} (g w^{\frac{1}{p}})(x)} \int_{B(x,t)} dy \frac{dt}{t^{n+1}}\, dx\right| \\ &\le C_m \int_{\mathbb{R}^n} \int_0^\infty \int_{B(y,t)} \left|F(x,t) \overline{(t^2 L^*)^m e^{-t\sqrt{L^*}} (g w^{\frac{1}{p}})(x)}\right| \frac{dx\,dt}{t^{n+1}}\, dy \\ &\lesssim \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)} \|Q_L(\bar{g} w^{\frac{1}{p}})\|_{T^{p'}(w^{1-p'})} \\ &\lesssim \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)} \|g w^{\frac{1}{p}}\|_{L^{p'}(w^{1-p'})} = \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)} \|g\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

Consequently, taking the supremum over all $g \in L^{p'}(\mathbb{R}^n)$ such that $\|g\|_{L^{p'}(\mathbb{R}^n)} \le 1$, we obtain that, for all $m \in \mathbb{N}$,

$$\|f\|_{L^p(w)} \lesssim \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)}. \tag{3.51}$$

Now note that, from the proof of Theorem 3.4, part(b), we have that $\|\mathcal{S}_{m,\mathrm{P}}f\|_{L^p(w)} \leq \|\mathcal{S}_{m,\mathrm{H}}f\|_{L^p(w)}$, for all $m \in \mathbb{N}$. This, (3.51), Theorems 3.3, 3.4, 3.23, and 3.29 yield that

$$\|f\|_{L^p(w)} \lesssim \|\mathcal{T} f\|_{L^p(w)},$$

where $\mathcal{T}$ is any of the operators in (1.26)-(1.31) or (1.32). This finishes the proof. $\square$

Finally, recall that, as we explained in Remark 1.42, if we consider $w \in A_\infty$ such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, for real operators it happens that $\widehat{p}_-(L) = r_w p_-(L)$. But for complex operators we do not know if the equality holds, we just know that $\widehat{p}_-(L) \leq r_w p_-(L)$. Hence, in view of the possibility that $\widehat{p}_-(L)$ and $r_w p_-(L)$ have different values, in the next result we improve the boundedness range of $\mathcal{S}_{\mathrm{H}}$; and with the same effort, due to the comparison results in Theorem 3.3, of others square functions associated with the heat semigroup.

Theorem 3.52. *Given $w \in A_\infty$ such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, for all $p \in (\widehat{p}_-(L), \infty)$, there hold:*

(a) *$\mathcal{S}_{\mathrm{H}}$ is bounded on $L^p(w)$.*

(b) *Given $m \in \mathbb{N}$, $\mathcal{S}_{m,\mathrm{H}}$, $\mathrm{G}_{m,\mathrm{H}}$, and $\mathcal{G}_{m,\mathrm{H}}$ are bounded on $L^p(w)$.*

If we further assume that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$ we have that, for $p \in (\widehat{p}_-(L), \infty)$, G_{H} and $\mathcal{G}_{\mathrm{H}}$ are bounded on $L^p(w)$ for all $p \in (\widehat{p}_-(L), \infty)$.

Proof. Note that $\mathcal{G}_{\mathrm{H}} \leq 2\mathcal{S}_{\mathrm{H}} + \mathrm{G}_{\mathrm{H}}$, then by Theorem 3.3, we just need to prove the theorem for $\mathcal{S}_{\mathrm{H}}$ and G_{H}. Let Q be $\mathcal{S}_{\mathrm{H}}$ or G_{H}

By [11, Theorem 2.4], to conclude our result, it is enough to prove that for every ball $B = B(x_B, r_B) \subset \mathbb{R}^n$

$$\left(\fint_{C_j(B)} |\mathcal{G}_{\mathrm{H}} B_{r_B} f(x)|^p dw\right)^{\frac{1}{p}} \lesssim g(j) \left(\fint_B |f(x)|^p dw\right)^{\frac{1}{p}}, \quad \text{for all} \quad j \geq 2; \tag{3.53}$$

and,

$$\left(\fint_{C_j(B)} |A_{r_B} f(x)|^q dw\right)^{\frac{1}{q}} \lesssim g(j) \left(\fint_B |f(x)|^p dw\right)^{\frac{1}{p}}, \quad \text{for all} \quad j \geq 1, \tag{3.54}$$

where $A_{r_B} := I - (I - e^{-r_B^2 L})^M$ and $B_{r_B} := I - A_{r_B}$, for some $M \in \mathbb{N}$ arbitrarily large, q is such that Q is bounded on $L^q(w)$, $f \in L^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp} f \subset B$, and $g(j)$ is such that $\sum_{j\geq 1} g(j) 2^{nr}$, for some $r > r_w$.

We start by taking $q \in \mathcal{W}_w(p_-(L), p_+(L))$ when $Q = \mathcal{S}_{\mathrm{H}}$ and $q \in \mathcal{W}_w(q_-(L), q_+(L))$ when $Q = \mathrm{G}_{\mathrm{H}}$, by Theorem 3.1, we know that, in any case, Q in bounded on $L^q(w)$. Besides, also by that result, we only need to consider the case $\widehat{p}_-(L) < p \leq r_w p_-(L)$ (we recall that $p_-(L) = q_-(L)$). Next, we fix p_0 so that $p_-(L) < p_0 < \min\{2, q\}$ and $w \in A_{q/p_0}$.

The proof of (3.54) follows by expanding the binomial and using that, for $1 \leq k \leq M$, $e^{-k r_B^2 L}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls, (see Section 1.3.2).

As for (3.53), first note that it is enough to prove

$$I := \left(\fint_{C_j(B)} \left(\int_0^\infty \int_{B(x,t)} |T_t B_{r_B} f(y)|^2 \frac{dy\, dt}{t^{n+1}}\right)^{\frac{p}{2}} dw\right)^{\frac{1}{p}} \lesssim g(j) \left(\fint_B |f(x)|^p dw\right)^{\frac{1}{p}},$$

for T_t being $t^2 L e^{-t^2 L}$ or $t\nabla_y e^{-t^2 L}$. Splitting the integral in t we have that

$$I \leq \left(\fint_{C_j(B)} \left(\int_0^{r_B} \int_{B(x,t)} |T_t B_{r_B} f(y)|^2 \frac{dy\, dt}{t^{n+1}}\right)^{\frac{p}{2}} dw\right)^{\frac{1}{p}}$$

$$+\left(\fint_{C_j(B)}\left(\int_{r_B}^{\infty}\int_{B(x,t)}|T_tB_{r_B}f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}} =: I_1+I_2. \quad (3.55)$$

In order to estimate I_2, consider $B_{r_B,t} := (e^{-t^2L}-e^{-(t^2+r_B^2)L})^M$. Then, changing the variable t into $t\sqrt{M+1} =: tC_M$ and applying that $\{T_t\}_{t>0}$ satisfies $L^{p_0}(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimates (see Section 1.3.1), we have

$$I_2 \lesssim \left(\fint_{C_j(B)}\left(\int_{\frac{r_B}{C_M}}^{\infty}\int_{B(x,tC_M)}|T_tB_{r_B,t}f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}$$

$$\lesssim \sum_{i\geq 1}e^{-c4^i}\left(\fint_{C_j(B)}\left(\int_{\frac{r_B}{C_M}}^{\infty}\left(\int_{B(x,2^{i+1}tC_M)}|B_{r_B,t}f(y)|^{p_0}\frac{dy}{t^n}\right)^{\frac{2}{p_0}}\frac{dt}{t}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}.$$

Besides, since $w\in A_{\frac{q}{p_0}}$, note that we have the following estimate for the integral in y:

$$\left(\int_{B(x,2^{i+1}C_Mt)}|B_{r_B,t}f(y)|^{p_0}\frac{dy}{t^n}\right)^{\frac{2}{p_0}} \quad (3.56)$$

$$\lesssim 2^{\frac{i2n}{p_0}}\left(\int_{B(x,2^{i+1}C_Mt)}|B_{r_B,t}f(y)|^q w(y)dy\right)^{\frac{2}{q}}\left(\fint_{B(x,2^{i+1}tC_M)}w(y)^{1-\left(\frac{q}{p_0}\right)'}dy\right)^{\frac{2}{q}\left(\frac{q}{p_0}-1\right)}(2^it)^{-\frac{2n}{q}}$$

$$\lesssim 2^{\frac{i2n}{p_0}}\left(\fint_{B(x,2^{i+1}C_Mt)}|B_{r_B,t}f(y)|^q dw\right)^{\frac{2}{q}}.$$

By (3.56), we can split I_2 as follows:

$$I_2 \lesssim \sum_{i=1}^{j-2}e^{-c4^i}2^{\frac{in}{p_0}}\left(\fint_{C_j(B)}\left(\int_{\frac{r_B}{C_M}}^{\infty}\left(\fint_{B(x,2^{i+1}tC_M)}|B_{r_B,t}f(y)|^q dw\right)^{\frac{2}{q}}\frac{dt}{t}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}$$

$$+e^{-c4^j}\sum_{i\geq j-1}e^{-c4^i}2^{\frac{in}{p_0}}\left(\fint_{C_j(B)}\left(\int_{\frac{r_B}{C_M}}^{\infty}\left(\fint_{B(x,2^{i+1}tC_M)}|B_{r_B,t}f(y)|^q dw\right)^{\frac{2}{q}}\frac{dt}{t}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}$$

$$=: \sum_{i=1}^{j-2}e^{-c4^i}I_{2i}^1+e^{-c4^j}\sum_{i\geq j-1}e^{-c4^i}I_{2i}^2.$$

The sum $\sum_{i=1}^{j-2}e^{-c4^i}I_{2i}^1$ only appears when $j\geq 3$. In this case, we split the integral in t and observe that for $x\in C_j(B)$, and $r_BC_M^{-1}<t<r_BC_M^{-1}2^{j-i-2}$, we have that $B(x,2^{i+1}C_Mt)\subset 2^{j+2}B\setminus 2^{j-1}B$; besides, for $1\leq i\leq j-2$ and $t\geq r_BC_M^{-1}2^{j-i-2}$, $B, B(x,2^{i+1}C_Mt)\subset B(x_B,2^{j+2}C_Mt)$. Then, applying (1.11), Proposition 1.43, and the fact that $tC_M\geq r_B$, we obtain

$$I_{2i}^1 \lesssim 2^{\frac{jn}{p_0}}\left(\int_{\frac{r_B}{C_M}}^{\frac{2^{j-i-2}r_B}{C_M}}\left(\frac{r_B}{t}\right)^{\frac{2n}{p_0}}\left(\sum_{l=j-1}^{j+1}\left(\fint_{C_l(B)}|B_{r_B,t}f(y)|^q dw\right)^{\frac{1}{q}}\right)^2\frac{dt}{t}\right)^{\frac{1}{2}}$$

$$+2^{\frac{jn}{p_0}}\left(\int_{\frac{2^{j-i-2}r_B}{C_M}}^{\infty}\left(\fint_{B(x_B,2^{j+2}C_Mt)}|B_{r_B,t}(f\mathbf{1}_{B(x_B,2^{j+2}C_Mt)})(y)|^q dw\right)^{\frac{2}{q}}\frac{dt}{t}\right)^{\frac{1}{2}}$$

$$\lesssim \frac{2^{j\left(\frac{n}{p_0}+\theta_2\right)}\|f\|_{L^p(w)}}{w(B)^{\frac{1}{p}}}\left(2^{j\theta_1}\left(\int_{\frac{r_B}{C_M}}^{\frac{2^{j-i-2}r_B}{C_M}}\left(\frac{r_B}{t}\right)^{\frac{2n}{p_0}+4M+2\theta_2}e^{-c\frac{4^j r_B^2}{t^2}}\frac{dt}{t}\right)^{\frac{1}{2}}+\left(\int_{\frac{2^{j-i-2}r_B}{C_M}}^{\infty}\left(\frac{r_B}{t}\right)^{4M}\frac{dt}{t}\right)^{\frac{1}{2}}\right)$$
$$\lesssim 2^{i2M}2^{-j\left(2M-\frac{n}{p_0}-\theta_1-\theta_2\right)}\left(\fint_B|f(y)|^p dw\right)^{\frac{1}{p}}.$$

The estimate of I_{2i}^2 follows applying Proposition 1.43 and the fact that for $x\in C_j(B)$, $j\geq 2$, $i\geq j-1$, and $tC_M\geq r_B$, we have that $B, B(x,2^{i+1}C_Mt)\subset B(x_B,2^{i+3}C_Mt)$

$$I_{2i}^2\lesssim 2^{\frac{in}{p_0}}\left(\int_{\frac{r_B}{C_M}}^{\infty}\left(\fint_{B(x_B,2^{i+3}C_Mt)}|B_{r_B,t}(f\mathbf{1}_{B(x_B,2^{i+3}C_Mt)})(y)|^q dw\right)^{\frac{2}{q}}\frac{dt}{t}\right)^{\frac{1}{2}}$$
$$\lesssim 2^{i\left(\frac{n}{p_0}+\theta_2\right)}\left(\int_{\frac{r_B}{C_M}}^{\infty}\left(\frac{r_B}{t}\right)^{4M}\frac{dt}{t}\right)^{\frac{1}{2}}\left(\fint_B|f(y)|^p dw\right)^{\frac{1}{p}}\lesssim 2^{i\left(\frac{n}{p_0}+\theta_2\right)}\left(\fint_B|f(y)|^p dw\right)^{\frac{1}{p}}.$$

Therefore, for all $j\geq 2$, we have

$$I_2\lesssim\left(2^{-j\left(2M-\frac{n}{p_0}-\theta_1-\theta_2\right)}+e^{-c4^j}\right)\left(\fint_B|f(y)|^p dw\right)^{\frac{1}{p}}. \tag{3.57}$$

Next, in order to estimate I_1, we expand the binomial. Then,

$$I_1\leq\left(\fint_{C_j(B)}\left(\int_0^{r_B}\int_{B(x,t)}|T_tf(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}$$
$$+\sum_{k=1}^{M}C_{k,M}\left(\fint_{C_j(B)}\left(\int_0^{r_B}\int_{B(x,t)}|T_te^{-kr_B^2L}f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}=:I_1^1+\sum_{k=1}^{M}C_{k,M}\mathcal{I}_k. \tag{3.58}$$

We first estimate I_1^1, noticing that $T_t=cT_{t/\sqrt{2}}e^{-\frac{t^2}{2}L}$, and applying the $L^{p_0}(\mathbb{R}^n)-L^2(\mathbb{R}^n)$ off-diagonal estimates satisfied by $T_{t/\sqrt{2}}$, we have

$$I_1^1\lesssim\sum_{i\geq 1}e^{-c4^i}\left(\fint_{C_j(B)}\left(\int_0^{r_B}\left(\int_{B(x,2^{i+1}t)}|e^{-\frac{t^2}{2}L}f(y)|^{p_0}\frac{dy}{t^n}\right)^{\frac{2}{p_0}}\frac{dt}{t}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}$$
$$\lesssim\sum_{i=1}^{j-2}e^{-c4^i}\left(\fint_{C_j(B)}\left(\int_0^{r_B}\left(\int_{B(x,2^{i+1}t)}|e^{-\frac{t^2}{2}L}f(y)|^{p_0}\frac{dy}{t^n}\right)^{\frac{2}{p_0}}\frac{dt}{t}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}$$
$$+e^{-c4^j}\sum_{i\geq j-1}e^{-c4^i}\left(\fint_{C_j(B)}\left(\int_0^{r_B}\left(\int_{B(x,2^{i+1}t)}|e^{-\frac{t^2}{2}L}f(y)|^{p_0}\frac{dy}{t^n}\right)^{\frac{2}{p_0}}\frac{dt}{t}\right)^{\frac{p}{2}}dw\right)^{\frac{1}{p}}$$
$$=:\sum_{i=1}^{j-2}e^{-c4^i}II_i+e^{-c4^j}\sum_{i\geq j-1}e^{-c4^i}III_i,$$

where the sum $\sum_{i=1}^{j-2}e^{-c4^i}II_i$ only appears if $j\geq 3$. Then, proceeding as in (3.56), and noticing that for $x\in C_j(B)$, $j\geq 3$, $1\leq i\leq j-2$, and $0<t<r_B$, we have that $B(x,2^{i+1}t)\subset 2^{j+2}B\setminus 2^{j-1}B$, applying the

$L^p(w) - L^q(w)$ off-diagonal estimates on balls satisfied by $e^{-\frac{t^2}{2}L}$ (see Proposition 1.41), we obtain that, for some constants $\theta_1, \theta_2 > 0$,

$$II_i \lesssim \left(\int_0^{r_B} \left(\frac{2^j r_B}{t} \right)^{\frac{2n}{p_0}} \left(\sum_{l=j-1}^{j+1} \left(\fint_{C_l(B)} |e^{-\frac{t^2}{2}L} f(y)|^q dw \right)^{\frac{1}{q}} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\lesssim e^{-c4^j} \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}} \left(\int_0^{r_B} \left(\frac{2^j r_B}{t} \right)^{\frac{2n}{p_0}+2\theta_2} e^{-c\frac{4^j r_B^2}{t^2}} \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim e^{-c4^j} \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}}.$$

Now we split III_i as follows

$$III_i \lesssim \left(\fint_{C_j(B)} \left(\int_0^{\frac{r_B}{2^{i+1}}} \left(\int_{B(x,2^{i+1}t)} |e^{-\frac{t^2}{2}L} f(y)|^{p_0} \frac{dy}{t^n} \right)^{\frac{2}{p_0}} \frac{dt}{t} \right)^{\frac{p}{2}} dw \right)^{\frac{1}{p}}$$

$$+ \left(\fint_{C_j(B)} \left(\int_{\frac{r_B}{2^{i+1}}}^{r_B} \left(\int_{B(x,2^{i+1}t)} |e^{-\frac{t^2}{2}L} f(y)|^{p_0} \frac{dy}{t^n} \right)^{\frac{2}{p_0}} \frac{dt}{t} \right)^{\frac{p}{2}} dw \right)^{\frac{1}{p}} =: III_i^1 + III_i^2.$$

Note that for $x \in C_j(B)$ and $0 < t < r_B/2^{i+1}$ we have that $B(x, 2^{i+1}t) \subset 2^{j+2}B \setminus 2^{j-1}B$. Then, if $j \geq 3$, the estimate of III_i^1 follows as the estimate of II_i. If $j = 2$, we write $B(x, 2^{i+1}t) \subset \bigcup_{l=2}^3 C_l(B) \cup (4B \setminus 2B)$ and proceed as in the estimate of II_i, applying [10, Lemma 6.5]. Hence, we obtain

$$III_i^1 \lesssim e^{-c4^j} \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}}.$$

In order to estimate III_i^2, we observe that for $x \in C_j(B)$, $j \geq 2$, $i \geq j-1$, and $r_B/2^{i+1} \leq t < r_B$, we have that $B, B(x, 2^{i+1}t) \subset 2^{i+3}B$. Thus, proceeding as before

$$III_i^2 \lesssim 2^{\frac{in}{p_0}} \left(\int_{\frac{r_B}{2^{i+1}}}^{r_B} \left(\frac{r_B}{t} \right)^{\frac{2n}{p_0}} \left(\fint_{2^{i+3}B} |e^{-\frac{t^2}{2}L}(f \mathbf{1}_{2^{i+3}B})(y)|^q dw \right)^{\frac{2}{q}} \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\lesssim \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}} \left(\int_{\frac{r_B}{2^{i+1}}}^{r_B} \left(\frac{2^i r_B}{t} \right)^{2\theta_2 + \frac{2n}{p_0}} \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim 2^{ic} \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}}.$$

Consequently, we conclude that

$$I_1^1 \lesssim e^{-c4^j} \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}}. \tag{3.59}$$

Let us now estimate $\mathcal{I}_k$. We shall use extrapolation to show that $\mathcal{I}_k \lesssim e^{-c4^j} \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}}$ for all $k \in \mathbb{N}$. To this end, we first show

that for every $w_0 \in RH_{\left(\frac{p_+(L)}{2}\right)'}$ if $T_t = t^2 L e^{-t^2 L}$ (or $w_0 \in RH_{\left(\frac{q_+(L)}{2}\right)'}$ if $T_t = t \nabla_y e^{-t^2 L}$), and $k \in \mathbb{N}$,

$$\int_{C_j(B)} \int_0^{r_B} \int_{B(x,t)} |T_t e^{-k r_B^2 L} f(y)|^2 \frac{dy\, dt}{t^{n+1}} w_0(x) dx$$

$$\lesssim \int_{2^{j+3}B \setminus 2^{j-2}B} \left(\sum_{i \geq 1} e^{-c4^i} \left(\int_{B(x,2^{i+1}r_B)} |e^{-\frac{k r_B^2}{2}L} f(y)|^{p_0} \frac{dy}{r_B^n} \right)^{\frac{1}{p_0}} \right)^2 w_0(x) dx. \tag{3.60}$$

Then, note that since $p \le r_w p_-(L) < q < \frac{p_+(L)}{s_w}$ (or $p \le r_w p_-(L) < q < \frac{q_+(L)}{s_w}$), we have that $w \in RH_{\left(\frac{p_+(L)}{p}\right)'}$ (or $w \in RH_{\left(\frac{q_+(L)}{p}\right)'}$). Hence, (3.60) and Theorem 1.46 part (*b*) (or part (*e*) if $q_+(L), p_+(L) = \infty$) imply that for all $k \in \mathbb{N}$,

$$w(2^{j+1}B)\mathcal{I}_k^p \lesssim \int_{2^{j+3}B\setminus 2^{j-2}B}\left(\sum_{i\ge1} e^{-c4^i}\left(\int_{B(x,2^{i+1}r_B)}|e^{-\frac{kr_B^2}{2}L}f(y)|^{p_0}\frac{dy}{r_B^n}\right)^{\frac{1}{p_0}}\right)^p w(x)dx =: II.$$

Thus, once proved (3.60), to estimate $\mathcal{I}_k$ we just need to consider II. Let us postpone the proof of (3.60) until later and continue with the estimate of $\mathcal{I}_k$. Since $w \in A_{\frac{q}{p_0}}$, proceeding as in (3.56), we have

$$II \lesssim \left(\sum_{i\ge1} e^{-c4^i}2^{\frac{in}{p_0}}\left(\int_{2^{j+3}B\setminus 2^{j-2}B}\left(\fint_{B(x,2^{i+1}r_B)}|e^{-\frac{kr_B^2}{2}L}f(y)|^q dw\right)^{\frac{p}{q}}dw\right)^{\frac{1}{p}}\right)^p.$$

For $2 \le j \le 4$, note that if $x \in 2^{j+3}B \setminus 2^{j-2}B$ then $B, B(x,2^{i+1}r_B) \subset 2^{i+7}B$. Hence, using (1.11) and the $L^p(w) - L^q(w)$ off-diagonal estimates on balls satisfied by $e^{-\frac{kr_B^2}{2}L}$, we get

$$II^{\frac{1}{p}} \lesssim \sum_{i\ge1} e^{-c4^i}w(B)^{\frac{1}{p}}\left(\fint_{2^{i+7}B}|e^{-\frac{kr_B^2}{2}L}(f\mathbf{1}_{2^{i+7}B})(y)|^q dw\right)^{\frac{1}{q}} \lesssim \sum_{i\ge1} e^{-c4^i}\left(\int_B |f(y)|^p dw\right)^{\frac{1}{p}} \lesssim \left(\int_B |f(y)|^p dw\right)^{\frac{1}{p}},$$

And, for $j \ge 5$, we proceed as before, but noticing this time that for $x \in 2^{j+3}B \setminus 2^{j-2}B$, if $1 \le i \le j-4$ then $B(x,2^{i+1}r_B) \subset 2^{j+4}B \setminus 2^{j-3}B$; and if $i \ge j-3$ $B, B(x,2^{i+1}r_B) \subset 2^{i+7}B$. Hence,

$$\begin{aligned}
II^{\frac{1}{p}} &\lesssim 2^{\frac{jn}{p_0}}w(2^{j+1}B)^{\frac{1}{p}}\sum_{i=1}^{j-4}e^{-c4^i}\sum_{l=j-3}^{j+3}\left(\fint_{C_l(B)}|e^{-\frac{kr_B^2}{2}L}f(y)|^q dw\right)^{\frac{1}{q}}\\
&\quad + e^{-c4^j}w(2^{j+1}B)^{\frac{1}{p}}\sum_{i\ge j-3}e^{-c4^i}\left(\fint_{2^{i+7}B}|e^{-\frac{kr_B^2}{2}L}(f\mathbf{1}_{2^{i+7}B})(y)|^q dw\right)^{\frac{1}{q}}\\
&\lesssim e^{-c4^j}w(2^{j+1}B)^{\frac{1}{p}}\left(\fint_B |f(y)|^p dw\right)^{\frac{1}{p}}.
\end{aligned}$$

Let us next prove (3.60). If $T_t = t^2Le^{-t^2L}$, for $w_0 \in RH_{\left(\frac{p_+(L)}{2}\right)'}$, if $p_+(L) < \infty$, we chose $2 < \widetilde{q} < p_+(L)$ so that $w_0 \in RH_{\left(\frac{\widetilde{q}}{2}\right)'}$; if $p_+(L) = \infty$, the condition $w_0 \in RH_{\left(\frac{p_+(L)}{2}\right)'}$ becomes $w_0 \in A_\infty$. In this case, we take $\widetilde{q}/2 > s_w$, consequently $w \in RH_{\left(\frac{\widetilde{q}}{2}\right)'}$ and $\widetilde{q}/2 > 1$. If $T_t = t\nabla_y e^{-t^2L}$, we do the same but replacing $p_+(L)$ with $q_+(L)$. Hence, by Proposition 2.64 and applying the $L^{p_0}(\mathbb{R}^n) - L^{\widetilde{q}}(\mathbb{R}^n)$ off-diagonal estimates satisfied by $T_{\sqrt{t^2+kr_B^2/2}}$, we have that, for $\eta = 2$ if $T_t = t\nabla_y e^{-t^2L}$ or $\eta = 4$ if $T_t = t^2Le^{-t^2L}$,

$$\begin{aligned}
&\int_{C_j(B)}\int_0^{r_B}\int_{B(x,t)}|T_te^{-kr_B^2L}f(y)|^2\frac{dy\,dt}{t^{n+1}}w_0(x)dx\\
&\lesssim \int_0^{r_B}\left(\frac{t}{r_B}\right)^\eta\int_{C_j(B)}\left(\int_{B(x,r_Bt/r_B)}|T_{\sqrt{t^2+kr_B^2/2}}e^{-\frac{kr_B^2}{2}L}f(y)|^{\widetilde{q}}\frac{dy}{t^n}\right)^{\frac{2}{\widetilde{q}}}w_0(x)dx\frac{dt}{t}\\
&\lesssim \int_0^{r_B}\left(\frac{t}{r_B}\right)^\eta\int_{2^{j+3}B\setminus 2^{j-2}B}\left(\int_{B(x,r_B)}|T_{\sqrt{t^2+kr_B^2/2}}e^{-\frac{kr_B^2}{2}L}f(y)|^{\widetilde{q}}\frac{dy}{r_B^n}\right)^{\frac{2}{\widetilde{q}}}w_0(x)dx\frac{dt}{t}\\
&\lesssim \int_0^{r_B}\left(\frac{t}{r_B}\right)^\eta\frac{dt}{t}\int_{2^{j+3}B\setminus 2^{j-2}B}\left(\sum_{i\ge1}e^{-c4^i}\left(\int_{B(x,2^{i+1}r_B)}|e^{-\frac{kr_B^2}{2}L}f(y)|^{p_0}\frac{dy}{r_B^n}\right)^{\frac{1}{p_0}}\right)^2 w_0(x)dx
\end{aligned}$$

$$\lesssim \int_{2^{j+3}B\setminus 2^{j-2}B} \left(\sum_{i\geq 1} e^{-c4^i} \left(\int_{B(x,2^{i+1}r_B)} |e^{-\frac{kr_B^2}{2}L} f(y)|^{p_0} \frac{dy}{r_B^n} \right)^{\frac{1}{p_0}} \right)^2 w_0(x) dx.$$

Therefore, we conclude that $\mathcal{I}_k \lesssim e^{-c4^j} \left(\fint_B |f(y)|^p dw \right)^{\frac{1}{p}}$, for all $k \in \mathbb{N}$. This, (3.55), (3.57) with $2M > n/p_0 + \frac{nq}{p_0} + \theta_1 + \theta_2$, (3.58), and (3.59), allow us to conclude the proof. □

Remark 3.61. *Note that, from the previous result and Theorem 3.4, in the case that $w \in A_\infty$ satisfying $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, we also improve the lower exponent of the range of p's where the conical square function associated with the Poisson semigroup* (1.29)-(1.31) *are bounded on $L^p(w)$. With the exception that in the case of* $\mathrm{G_P}$ *and* $\mathcal{G}_{\mathrm{P}}$*, we need to assume further that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$.*

Chapter 4

WEIGHTED HARDY SPACES

The study of Hardy spaces began in the early 1900s in the context of Fourier series and complex analysis in one variable. It was not until 1960 when the theory in $\mathbb{R}^n$ started developing by E.M. Stein and G. Weiss ([76]). A few years later R.R. Coifman in [29] and R.H. Latter in [65] gave an atomic decomposition of the Hardy spaces H^p, $0 < p \leq 1$. This atomic decomposition turns out to be a very important tool when studying the boundedness of some singular integral operators, since in most cases checking the action of the operator in question on these simpler elements (atoms) suffices to conclude its boundedness in the corresponding Hardy space.

Another stage in the progress of the theory of Hardy spaces was done by J. García-Cuerva in [48] (see also [78]) when he considered $\mathbb{R}^n$ with the measure given by a Muckenhoupt weight. These spaces were called weighted Hardy spaces, and among other contributions, he also characterized them using an atomic decomposition.

In general, the development of the theory of Hardy spaces has contributed to give us a better understanding of some other topics as in the theory of singular integrals operators, maximal functions, multiplier operators, etc. However, there are some operators that escape from the theory of these classical Hardy spaces. These are, for example, the operators associated with a second order divergence form elliptic operator L, such as the conical square functions and non-tangential maximal functions defined by the heat and Poisson semigroups generated by the operator L, (see (1.26)-(1.31) and (1.32) for the precise definitions of these operators).

The theory of Hardy spaces associated with elliptic operators L was initiated by P. Auscher, X.T. Duong, and A. McIntosh in an unpublished work, [5]. Besides, P. Auscher and E. Russ in [17] considered the case on which the heat kernel associated with L is smooth and satisfies pointwise Gaussian bounds, this occurs for instance for real symmetric operators. There, among other things, it was shown that the corresponding Hardy space associated with L agrees with the classical Hardy space.

In the setting of Riemannian manifolds satisfying the doubling volume property, Hardy spaces associated with the Laplace-Beltrami operator are introduced in [12] by P. Auscher, A. McIntosh and E. Russ and it is shown that they admit several characterizations. Simultaneously, in the Euclidean setting, the study of Hardy spaces related to the conical square functions and non-tangential maximal functions associated with the heat and Poisson semigroups generated by divergence form elliptic operators L was taken by S. Hofmann and S. Mayboroda in [55], for $p = 1$. The new point was that only a form of decay weaker than pointwise bounds and satisfied in many occurrences was enough to develop a theory. This was followed later on by a second article of S. Hofmann, S. Mayboroda, and A. McIntosh [56], for a general p and, at the same time, by an article of R. Jiang and D. Yang [61]. They gave a molecular decomposition of those spaces and duality results. A natural line of study in the context of these new Hardy spaces is the development of a weighted theory for them, as J. García-Cuerva did in the classical setting. Some interesting progress has been done in this regard by T.A. Bui, J. Cao, L.D. Ky, D. Yang, and S. Yang in [23, 22]. The results obtained in [22] in the particular case $\varphi(x, t) := tw(x)$, where w is a Muckenhoupt weight, give characterizations of the weighted Hardy spaces that, however, only recover part of the results obtained in the unweighted case by simply taking $w = 1$.

In this chapter we define the weighted Hardy spaces associated with the conical square functions considered in (1.26)-(1.31), the non-tangential maximal functions defined in (1.32), and the Riesz transform (1.33). In the case of considering weighted Hardy spaces associated with an operator $\mathcal{T}$, that is, $H^p_{\mathcal{T}}(w)$, $0 < p \le 1$,where $\mathcal{T}$ is any conical square function or a non-tangential maximal function, we obtain a molecular characterization of them. This is particularly useful to prove different properties of these spaces as happens in the classical setting and in the context of second order divergence form elliptic operators. Moreover, we shall show that the spaces $H^p_{\mathcal{T}}(w)$ are isomorphic to $L^p(w)$ for $p \in \mathcal{W}_w(p_-(L), p_+(L))$. In the case that $\mathcal{T}$ is the Riesz transform, we characterize the corresponding Hardy spaces through the Hardy space associated with $\mathcal{S}_{\mathrm{H}}$, for a certain range of p's. We also remark that if we consider the weight equal to one we fully recover the results obtained in the unweighted case in [55] and [56].

4.1 Definitions

For $w \in A_\infty$ such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, $q \in \mathcal{W}_w(p_-(L), p_+(L))$, and $0 < p < \infty$. We define the weighted Hardy spaces associated with operators and the molecular weighted Hardy space.

4.1.1 Weighted Hardy spaces associated with operators

Definition 4.1. *Given a sublinear operator $\mathcal{T}$ acting on functions of $L^q(w)$ we define the weighted Hardy space $H^p_{\mathcal{T},q}(w)$ as the completion of the set*

$$\mathbb{H}^p_{\mathcal{T},q}(w) := \left\{ f \in L^q(w) : \mathcal{T} f \in L^p(w) \right\}, \tag{4.2}$$

with respect to the quasi-norm

$$\|f\|_{\mathbb{H}^p_{\mathcal{T},q}(w)} := \|\mathcal{T} f\|_{L^q(w)}. \tag{4.3}$$

In our results $\mathcal{T}$ will be any of the square functions in (1.26)–(1.31), or a non-tangential maximal functions in (1.32).

Definition 4.4. *Given $w \in A_\infty$ such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$, $q \in \mathcal{W}_w(q_-(L), q_+(L))$, we define the weighted Hardy space associated with the Riesz transform $H^p_{\nabla L^{-1/2},q}(w)$, as the completion of the set*

$$\mathbb{H}^p_{\nabla L^{-1/2},q}(w) := \{ f \in L^q(w) : \|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)} < \infty \},$$

with respect to the quasi-norm $\|f\|_{\mathbb{H}^p_{\nabla L^{-1/2}}} := \|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)}$.

Remark 4.5. *In [56], where the unweighted case was considered, the Hardy spaces are defined taking the completion of a set of functions in $L^2(\mathbb{R}^n)$. Here we take functions in $L^q(w)$, where $q \in \mathcal{W}_w(p_-(L), p_+(L))$ because we don't know whether 2 is in $\mathcal{W}_w(p_-(L), p_+(L))$ or not. In any case, we shall show that for $0 < p \le 1$ or $p \in \mathcal{W}_w(p_-(L), p_+(L))$ this choice of q is irrelevant since all the spaces $H^p_{\mathcal{T},q}(w)$ are isomorphic for all $q \in \mathcal{W}_w(p_-(L), p_+(L))$.*

4.1.2 Molecular weighted Hardy spaces

Recall that for any cube $Q \subset \mathbb{R}^n$ we set

$$\ell(Q_i) := 2^{i+1}\ell(Q), \quad \forall\, i \ge 1, \quad C_1(Q) := 4Q \quad \text{and,} \quad C_i(Q) := 2^{i+1}Q \backslash 2^i Q \quad \forall\, i \ge 2, \tag{4.6}$$

where $\ell(Q)$ denotes the side length of Q.

We next define the notion of molecules and molecular representation. These objects are a weighted version of those defined in [56] in the unweighted case.

Definition 4.7. *Apart from the conditions stated at the beginning of the section for w, q, and p, let us take $\varepsilon > 0$ and $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$.*

(*a*) ***Molecules:*** *We say that a function $\mathfrak{m} \in L^q(w)$ (belonging to the range of L^k in $L^q(w)$, $0 \le k \le M$,) is a $(w, q, p, \varepsilon, M)$-molecule if, for some cube $Q \subset \mathbb{R}^n$, $\mathfrak{m}$ satisfies*

$$\|\mathfrak{m}\|_{mol,w} := \sum_{i\ge1} 2^{i\varepsilon} w(2^{i+1}Q)^{\frac{1}{p}-\frac{1}{q}} \sum_{k=0}^{M} \left\| \left((\ell(Q)^2L)^{-k}\mathfrak{m}\right) \mathbf{1}_{C_i(Q)} \right\|_{L^q(w)} < 1.$$

Henceforth, we refer to the previous expression as the molecular w-norm of $\mathfrak{m}$. Besides, any cube Q satisfying that expression is called a cube associated with $\mathfrak{m}$.

Note that if $\mathfrak{m}$ is a $(w, q, p, \varepsilon, M)$-molecule, for all associated cubes Q:

$$\left\| \left((\ell(Q)^2L)^{-k}\mathfrak{m}\right) \mathbf{1}_{C_i(Q)} \right\|_{L^q(w)} \le 2^{-i\varepsilon} w(2^{i+1}Q)^{\frac{1}{q}-\frac{1}{p}} \quad i = 1, 2, \ldots; \; k = 0, 1, \ldots, M. \tag{4.8}$$

(*b*) ***Molecular representation:*** *For any function f, we say that $\sum_{i\in\mathbb{N}} \lambda_i \mathfrak{m}_i$ is a $(w, q, p, \varepsilon, M)$-representation of f, if the following conditions are satisfied:*

(*i*) $\{\lambda_i\}_{i\in\mathbb{N}} \in \ell^p$.

(*ii*) *For every $i \in \mathbb{N}$, $\mathfrak{m}_i$ is a $(w, q, p, \varepsilon, M)$-molecule.*

(*iii*) $f = \sum_{i\in\mathbb{N}} \lambda_i \mathfrak{m}_i$ *in $L^q(w)$.*

Using the properties of the weight w, we can obtain boundedness for the norm of $\left((\ell(Q)^2L)^{-k}\mathfrak{m}\right) \mathbf{1}_{C_i(Q)}$ in Lebesgue spaces instead of in weighted ones. That is:

Lemma 4.9. *Given $p_0 < q$, $0 < p < \infty$, $w \in A_{\frac{q}{p_0}}$, $\varepsilon > 0$, and $M \in \mathbb{N}$. Let $\mathfrak{m}$ be a $(w, q, p, \varepsilon, M)$-molecule and let Q be a cube associated with $\mathfrak{m}$. For every $i \ge 1$ and every $0 \le k \le M$, $k \in \mathbb{N}_0$, there holds*

$$\left\| \left((\ell(Q)^2L)^{-k}\mathfrak{m}\right) \mathbf{1}_{C_i}(Q) \right\|_{L^{p_0}(\mathbb{R}^n)} \lesssim 2^{-i\varepsilon} w(2^{i+1}Q)^{-\frac{1}{p}} |2^{i+1}Q|^{\frac{1}{p_0}}.$$

Proof. First of all, recall that if $\mathfrak{m}$ is a $(w, q, p, \varepsilon, M)$-molecule, in particular we have (4.8). This, Hölder's inequality, and the fact that $w \in A_{\frac{q}{p_0}}$ imply

$$\begin{aligned}
&\left\| \left((\ell(Q)^2L)^{-k}\mathfrak{m}\right) \mathbf{1}_{C_i(Q)} \right\|_{L^{p_0}(\mathbb{R}^n)} \\
&\le \left(\int_{C_i(Q)} |(\ell(Q)^2L)^{-k}\mathfrak{m}(y)|^q w(y)\, dy\right)^{\frac{1}{q}} \left(\fint_{2^{i+1}Q} w(y)^{1-\left(\frac{q}{p_0}\right)'} dy\right)^{\frac{1}{q}\left(\frac{q}{p_0}-1\right)} |2^{i+1}Q|^{\frac{1}{p_0}-\frac{1}{q}} \\
&\lesssim \left(\int_{C_i(Q)} |(\ell(Q)^2L)^{-k}\mathfrak{m}(y)|^q w(y)\, dy\right)^{\frac{1}{q}} \left(\fint_{2^{i+1}Q} w(y)\, dy\right)^{-\frac{1}{q}} |2^{i+1}Q|^{\frac{1}{p_0}-\frac{1}{q}} \\
&\lesssim 2^{-i\varepsilon} w(2^{i+1}Q)^{-\frac{1}{p}} |2^{i+1}Q|^{\frac{1}{p_0}}.
\end{aligned}$$

□

We finally define the molecular weighted Hardy spaces.

Definition 4.10. *Let w, q, p, ε, and M be as in the previous definition, we define the molecular weighted Hardy space $H^p_{L,q,\varepsilon,M}(w)$ as the completion of the set*

$$\mathbb{H}^p_{L,q,\varepsilon,M}(w) := \left\{ f = \sum_{i=1}^{\infty} \lambda_i \mathfrak{m}_i : \sum_{i=1}^{\infty} \lambda_i \mathfrak{m}_i \text{ is a } (w, q, p, \varepsilon, M)\text{-representation of } f \right\},$$

with respect to the quasi-norm,

$$\|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)} := \inf\left\{\left(\sum_{i=1}^{\infty}|\lambda_i|^p\right)^{\frac{1}{p}} : \sum_{i=1}^{\infty}\lambda_i\mathfrak{m}_i \text{ is a } (w,q,p,\varepsilon,M)\text{-representation of } f\right\}.$$

Remark 4.11. *Although we shall just show molecular characterization for weighted Hardy spaces in the range* $0 < p \leq 1$*, we have given the definition of the molecular weighted Hardy spaces for all* $0 < p < \infty$*. This is because we can always obtain a molecular decomposition of functions* $f \in \mathbb{H}^p_{\mathcal{T},q}$*. This is easily seen by following the proof of the molecular decomposition of functions in* $\mathbb{H}^p_{\mathcal{T},q}$ *and noticing that there is no restriction over p. In particular, we have that* $\mathbb{H}^p_{\mathcal{T},q}(w) \subset \mathbb{H}^p_{L,q,\varepsilon,M}(w)$*, with* $\|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)} \lesssim \|f\|_{\mathbb{H}^p_{\mathcal{T},q}(w)}$ *for all* $0 < p < \infty$*.*

Remark 4.12. *We shall show below that, for* $0 < p \leq 1$*, the Hardy space* $H^p_{L,q,\varepsilon,M}(w)$ *does not depend on the choice of the allowable parameters q, ε, and M. Hence, at this point, it is convenient for us to make a choice of these parameters and define the weighted Hardy space as the one associated with this choice.*

From now on for every $w \in A_\infty$ *we fix* $q_0 \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \leq 1$, $\varepsilon_0 > 0$*, and* $M_0 \in \mathbb{N}$ *such that* $M_0 > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$*, and set* $H^p_L(w) := H^p_{L,q_0,\varepsilon_0,M_0}(w)$*.*

4.2 Interpolation of $H^p_{\mathcal{S}_{\mathrm{H}}}(w)$

Recall that in Theorem 2.76 we gave an interpolation result regarding weighed tent spaces. From that result and after showing that Hardy spaces are retracts of tent spaces, we also obtain complex interpolation for Hardy spaces.

Theorem 4.13. *Given* $w \in A_\infty$ *and* $q \in \mathcal{W}_w(p_-(L), p_+(L))$*. Suppose* $1 \leq p_0 < p_1 < \frac{p_+(L)^{1/2,*}}{s_w}$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*,* $0 < \theta < 1$*. Then*

$$[H^{p_0}_{\mathcal{S}_{\mathrm{H}},q}(w), H^{p_1}_{\mathcal{S}_{\mathrm{H}},q}(w)]_\theta = H^p_{\mathcal{S}_{\mathrm{H}},q}(w).$$

Proof. As we explained above, in view of Theorem 2.76, it is enough to show that Hardy spaces are retracts of weighted tent spaces, i.e., that there exists an operator from any tent space to the corresponding Hardy space having a right inverse.

Fix $q \in \mathcal{W}_w(p_-(L), p_+(L))$, and note that for a function $f \in L^q(w)$ and $m \in \mathbb{N}$, we have the following Calderón reproducing formula of f,

$$f = C_m \int_0^\infty \left((t^2L)^m e^{-t^2L}\right)^2 f\frac{dt}{t} \tag{4.14}$$

in $L^q(w)$, where C_m is a positive constant. This equality follows from the fact that, for all $w \in A_\infty$ and $p \in \mathcal{W}_w(p_-(L), p_+(L))$, the vertical square function defined by $(t^2L)^m e^{-t^2L}$ is bounded on $L^p(w)$, (see [11]), and by a similar explanation to that of Remark 3.49.

Besides, if we define for each $(y,t) \in \mathbb{R}^{n+1}_+$, $\mathcal{F}(y,t) := (t^2L)^m e^{-t^2L} f(y)$ and the operator $Q_{L,m}f := \mathcal{F}$ acting over functions in $L^q(w)$, by Theorem 3.3,

$$\|Q_{L,m}f\|_{T^p(w)} = \|\mathcal{S}_{m,\mathrm{H}}f\|_{L^p(w)} \leq \|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)},$$

then $Q_{L,m}$ is bounded from $\mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$ to $T^p(w)$, for all $q \in \mathcal{W}_w(p_-(L), p_+(L))$ and $1 \leq p < \infty$. Thus, by the definition of $H^p_{\mathcal{S}_{\mathrm{H}},q}(w)$, it can be extended to a bounded operator, denoted by $\mathfrak{Q}_{L,m}$ from $H^p_{\mathcal{S}_{\mathrm{H}},q}(w)$ to $T^p(w)$. Similarly if we consider $Q_{L^*,m}$, defined for all functions $f \in L^{q'}(w^{1-q'})$ by $Q_{L^*,m}f(y,t) := (t^2L^*)^m e^{-t^2L^*} f(y)$, for all $(y,t) \in \mathbb{R}^{n+1}_+$. Again by Theorem 3.3, by [9, Lemma 4.4], and since $p_\pm(L^*) = p_\mp(L)'$, see [3], we have that

$\mathcal{Q}_{L^*,m} : L^{q'}(w^{1-q'}) \to T^{q'}(w^{1-q'})$ for all $q' \in \mathcal{W}_{w^{1-q'}}(p_-(L^*), p_+(L^*))$. Moreover, for all $F \in T^2(\mathbb{R}^n)$, its adjoint operator $(\mathcal{Q}_{L^*,m})^*$, has the following representation

$$(\mathcal{Q}_{L^*,m})^* F(y) = \int_0^\infty (t^2L)^m e^{-t^2L} F(y,t)\frac{dt}{t}. \tag{4.15}$$

Then since for all $F \in T^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}\left|\int_0^\infty (t^2L)^m e^{-t^2L} F(y,t)\frac{dt}{t}\right| &= \left|\int_{\mathbb{R}^n}\int_0^\infty F(y,t)\overline{(t^2L^*)^m e^{-t^2L^*} g(y)}\int_{B(y,t)} dx\frac{dt\,dy}{t^{n+1}}\right| \\ &\leq \int_{\mathbb{R}^n} |||F|||_{\Gamma(x)} |||(t^2L^*)^m e^{-t^2L^*} g|||_{\Gamma(x)} dx,\end{aligned}$$

where $|||F|||_{\Gamma(x)} = \left(\iint_{\Gamma(x)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}$. By a density argument, we conclude that $(\mathcal{Q}_{L^*,m})^*$ has a bounded extension, denoted by $\widetilde{\mathcal{Q}}_{L,m}$, from $T^q(w)$ to $L^q(w)$, for all $q \in \mathcal{W}_w(p_-(L), p_+(L))$. As we explain in Remark 3.49, we also have the expression (4.15) for functions $F \in T^q(w)$ and $q \in \mathcal{W}_w(p_-(L), p_+(L))$, understanding that, by abuse of notation, in this case L is the infinitesimal generator of e^{-tL} on $L^q(w)$ (see [11, Remark 3.5]).

Besides we shall show that for all functions $F \in T^p(w) \cap T^q(w)$ and $m \in \mathbb{N}$ big enough,

$$\|\mathcal{S}_{\mathrm{H}}\widetilde{\mathcal{Q}}_{L,m}F\|_{L^p(w)} \lesssim \|F\|_{T^p(w)}, \quad \text{for all} \quad 1 \leq p < \frac{p_+(L)^{1/2,*}}{s_w}. \tag{4.16}$$

Assuming this, since $T^p(w)\cap T^q(w)$ is dense in $T^p(w)$, $\widetilde{\mathcal{Q}}_{L,m|_{T^p(w)}}$ can be extended to a bounded operator, denoted by $\mathfrak{Q}_{L,m}$, from $T^p(w)$ to $H^p_{\mathcal{S}_{\mathrm{H}},q}(w)$. Then, by (4.14), we have that $C_m\widetilde{\mathfrak{Q}}_{L,m} \circ \mathfrak{Q}_{L,m} = I$ in $\mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$, and by density in $H^p_{\mathcal{S}_{\mathrm{H}},q}(w)$. Hence, for $w \in A_\infty$, $1 \leq p < \frac{p_+(L)^{1/2,*}}{s_w}$, and $q \in \mathcal{W}_w(p_-(L), p_+(L))$ the Hardy spaces $H^p_{\mathcal{S}_{\mathrm{H}},q}(w)$ are retracts of the tent spaces $T^p(w)$.

Then to conclude the proof it just remains to show (4.16). Applying Minkowski's integral inequality we obtain

$$\begin{aligned}\|\mathcal{S}_{\mathrm{H}}\widetilde{\mathcal{Q}}_{L,m}F\|_{L^p(w)} &= \left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,t)}|t^2Le^{-t^2L}\widetilde{\mathcal{Q}}_LF(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}}w(x)dx\right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\int_0^\infty\left(\int_{B(x,t)}|t^2Le^{-t^2L}(s^2L)^me^{-s^2L}F(y,s)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{p}{2}}w(x)dx\right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\int_0^t\left(\int_{B(x,t)}|t^2Le^{-t^2L}(s^2L)^me^{-s^2L}F(y,s)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{p}{2}}w(x)dx\right)^{\frac{1}{p}} \\ &+ \left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\int_t^\infty\left(\int_{B(x,t)}|t^2Le^{-t^2L}(s^2L)^me^{-s^2L}F(y,s)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{p}{2}}w(x)dx\right)^{\frac{1}{p}} =: I + II.\end{aligned}$$

We first show by extrapolation that $II \lesssim \|F\|_{T^p(w)}$, for every p as in (4.16) and every $m \in \mathbb{N}$. To this end, in view of Theorem 1.46, part (b), (or part (e) if $p_+(L)^{1/2,*} = \infty$), it is enough to consider the case $p = 2$ and $w \in RH_{\left(\frac{p_+(L)^{1/2,*}}{2}\right)'}$. That is, to prove that, for all $w \in RH_{\left(\frac{p_+(L)^{1/2,*}}{2}\right)'}$ and all $m \in \mathbb{N}$,

$$III := \left(\int_{\mathbb{R}^n}\int_0^\infty\left(\int_t^\infty\left(\int_{B(x,t)}|t^2Le^{-t^2L}(s^2L)^me^{-s^2L}F(y,s)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}w(x)dx\right)^{\frac{1}{2}} \lesssim \|F\|_{T^2(w)}.$$

Under this assumption, note that we can find q_0 and r so that $2 < q_0 < p_+(L)$, $\frac{q_0}{2} \leq r < \infty$, $w \in RH_{r'}$, and

$$2 + \frac{n}{2r} - \frac{n}{q_0} > 0. \tag{4.17}$$

Indeed, if $n > 2p_+(L)$, since $w \in RH_{\left(\frac{p_+(L)^{1/2,*}}{2}\right)'}$, we have that $s_w < \frac{np_+(L)}{2(n-2p_+(L))}$. Therefore, there exist $\varepsilon_1 > 0$ small enough and $2 < q_0 < p_+(L)$ close enough to $p_+(L)$ so that

$$s_w < \frac{nq_0}{2(1+\varepsilon_1)(n-2q_0)}.$$

Besides, there exists $\varepsilon_2 > 0$ so that

$$q_0 < \frac{nq_0}{(1+\varepsilon_2)(n-2q_0)}.$$

Hence, taking $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$ and $r := \frac{nq_0}{2(1+\varepsilon_0)(n-2q_0)}$, we have that $2 < q_0 < p_+(L)$, $q_0/2 \leq r < \infty$, and $w \in RH_{r'}$. Moreover

$$2 + \frac{n}{2r} - \frac{n}{q_0} = \varepsilon_0 \left(\frac{n}{q_0} - 2\right) > \varepsilon_0 \left(\frac{n}{p_+(L)} - 2\right) > 0.$$

If now we consider $n \leq 2p_+(L)$, we have that $p_+(L)^{1/2,*} = \infty$. Then, note that the assumption $w \in RH_{\left(\frac{p_+(L)^{1/2,*}}{2}\right)'}$ becomes $w \in A_\infty$. Hence, we fix $r > s_w$, and q_0 satisfying $\max\left\{2, \frac{2rp_+(L)}{p_+(L)+2r}\right\} < q_0 < \min\{p_+(L), 2r\}$ if $p_+(L) < \infty$ and $q_0 = 2r$ if $p_+(L) = \infty$. Therefore, we have that $2 < q_0 < p_+(L)$, $q_0/2 \leq r < \infty$, and $w \in RH_{r'}$. Besides,

$$2 + \frac{n}{2r} - \frac{n}{q_0} > 2 - \frac{n}{p_+(L)} \geq 0.$$

Keeping these choices in mind, we apply the fact that $\{e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^2)$, change the variable s into st, and apply Jensen's inequality and Minkowski's integral inequality. Then, we have

$$\begin{aligned}
III &\leq \left(\int_{\mathbb{R}^n}\int_0^\infty \left(\int_t^\infty \frac{t^2}{s^2}\left(\int_{B(x,t)} |e^{-t^2L}(s^2L)^{m+1}e^{-s^2L}F(y,s)|^2 dy\right)^{\frac12}\frac{ds}{s}\right)^2 \frac{dt}{t^{n+1}} w(x)dx\right)^{\frac12} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j}\left(\int_{\mathbb{R}^n}\int_0^\infty \left(\int_t^\infty \frac{t^2}{s^2}\left(\int_{B(x,2^{j+1}t)} |(s^2L)^{m+1}e^{-s^2L}F(y,s)|^2 dy\right)^{\frac12}\frac{ds}{s}\right)^2 \frac{dt}{t^{n+1}} w(x)dx\right)^{\frac12} \\
&= \sum_{j\geq 1} e^{-c4^j}\left(\int_{\mathbb{R}^n}\int_0^\infty \left(\int_1^\infty s^{-2}\left(\int_{B(x,2^{j+1}t)} |((st)^2L)^{m+1}e^{-(st)^2L}F(y,st)|^2 dy\right)^{\frac12}\frac{ds}{s}\right)^2 \frac{dt}{t^{n+1}} w(x)dx\right)^{\frac12} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j}2^{jc_n}\int_1^\infty s^{-2}\left(\int_{\mathbb{R}^n}\int_0^\infty \left(\int_{B(x,2^{j+1}t)} |((st)^2L)^{m+1}e^{-(st)^2L}F(y,st)|^{q_0} \frac{dy}{t^n}\right)^{\frac{2}{q_0}} \frac{dt}{t} w(x)dx\right)^{\frac12}\frac{ds}{s}.
\end{aligned}$$

Now, consider for every $s \geq 1$

$$\mathcal{J}(x,s) := \left(\int_0^\infty \left(\int_{B(x,2^{j+1}t)} |((st)^2L)^{m+1}e^{-(st)^2L}F(y,st)|^{q_0}\frac{dy}{t^n}\right)^{\frac{2}{q_0}}\frac{dt}{t}\right)^{\frac12}.$$

Applying Fubini's theorem, Proposition 2.61, changing the variable t into t/s, the fact that $\{(t^2L)^{m+1}e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^{q_0})$, and Proposition 2.43, and recalling our choices of q_0 and r, we obtain that, for $r_0 > r_w$ and every $s \geq 1$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{J}(x,s)^2 w(x)dx &\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left(\int_{B(x,2^{j+1}st/s)} |((st)^2L)^{m+1}e^{-(st)^2L}F(y,st)|^{q_0}\frac{dy}{t^n}\right)^{\frac{2}{q_0}} w(x)dx\frac{dt}{t} \\
&\lesssim s^{-\frac{n}{r}} \int_0^\infty \int_{\mathbb{R}^n} \left(\int_{B(x,2^{j+1}st)} |((st)^2L)^{m+1}e^{-(st)^2L}F(y,st)|^{q_0}\frac{dy}{t^n}\right)^{\frac{2}{q_0}} w(x)dx\frac{dt}{t} \\
&\approx s^{-\frac{n}{r}+\frac{2n}{q_0}} \int_{\mathbb{R}^n}\int_0^\infty \left(\int_{B(x,2^{j+1}t)} |(t^2L)^{m+1}e^{-t^2L}F(y,t)|^{q_0}\frac{dy}{t^n}\right)^{\frac{2}{q_0}} \frac{dt}{t}w(x)dx \\
&\lesssim s^{-\frac{n}{r}+\frac{2n}{q_0}} \sum_{l\geq1} e^{-c4^l} \int_{\mathbb{R}^n}\int_0^\infty \int_{B(x,2^{j+l+2}t)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \\
&\lesssim 2^{\frac{jnr_0}{2}} s^{-\frac{n}{r}+\frac{2n}{q_0}} \|F\|^2_{T^2(w)}
\end{aligned}$$

Hence, by (4.17), we have

$$III \lesssim \sum_{j\geq1} e^{-c4^j}\int_1^\infty s^{-2}\left(\int_{\mathbb{R}^n}\mathcal{J}(x,s)^2w(x)dx\right)^{\frac{1}{2}}\frac{ds}{s} \lesssim \sum_{j\geq1} e^{-c4^j}\int_1^\infty s^{-2-\frac{n}{2r}+\frac{n}{q_0}}\frac{ds}{s}\|F\|_{T^2(w)} \lesssim \|F\|_{T^2(w)}$$

which, as we observed above, implies that $II \lesssim \|F\|_{T^p(w)}$, for all $1 \leq p < \frac{p_+(L)^{1/2,*}}{s_w}$ and all $m \in \mathbb{N}$.

Next, in order to estimate I we apply the fact that $\{(t^2L)^{m+1}e^{-t^2L}\}_{t>0}$, $\{e^{-s^2L}\}_{s>0} \in \mathcal{F}_\infty(L^2 \to L^2)$, and Lemma 1.35 recalling that, in this case, $s < t$. Then,

$$\begin{aligned}
\left(\int_{B(x,t)} |t^2Le^{-t^2L}(s^2L)^m e^{-s^2L}F(y,s)|^2dy\right)^{\frac{1}{2}} &= \left(\frac{s^2}{t^2}\right)^m \left(\int_{B(x,t)} |(t^2L)^{m+1}e^{-t^2L}e^{-s^2L}F(y,s)|^2dy\right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{s^2}{t^2}\right)^m \sum_{j\geq1} e^{-c4^j}\left(\int_{B(x,2^{j+1}t)} |F(y,s)|^2dy\right)^{\frac{1}{2}}.
\end{aligned}$$

Using this, changing the variable s into st, applying Minkowski's integral inequality twice, changing the variable t into t/s, applying Proposition 2.43, and taking $2m > \frac{nr_0}{p} - \frac{n}{2}$, where $r_0 > \max\{p/2, r_w\}$, we obtain that

$$\begin{aligned}
I &\lesssim \sum_{j\geq1} e^{-c4^j}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\int_0^t\left(\frac{s^2}{t^2}\right)^m\left(\int_{B(x,2^{j+1}t)}|F(y,s)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{p}{2}}w(x)dx\right)^{\frac{1}{p}} \\
&= \sum_{j\geq1} e^{-c4^j}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\int_0^1 s^{2m}\left(\int_{B(x,2^{j+1}t)}|F(y,st)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{p}{2}}w(x)dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\geq1} e^{-c4^j}\left(\int_{\mathbb{R}^n}\left(\int_0^1 s^{2m}\left(\int_0^\infty\int_{B(x,2^{j+1}t)}|F(y,st)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}\frac{ds}{s}\right)^p w(x)dx\right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\geq1} e^{-c4^j}\int_0^1 s^{2m}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,2^{j+1}t)}|F(y,st)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p}{2}}w(x)dx\right)^{\frac{1}{p}}\frac{ds}{s}
\end{aligned}$$

$$= \sum_{j\geq 1} e^{-c4^j} \int_0^1 s^{2m+\frac{n}{2}} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,2^{j+1}t/s)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p}{2}} w(x)dx \right)^{\frac{1}{p}} \frac{ds}{s}$$

$$\lesssim \sum_{j\geq 1} 2^{j\frac{nr_0}{p}} e^{-c4^j} \int_0^1 s^{2m+\frac{n}{2}-\frac{nr_0}{p}} \frac{ds}{s} \|F\|_{T^p(w)} \lesssim \|F\|_{T^p(w)},$$

which finishes the proof. □

4.3 Characterization of $H^p_L(w)$, $0 < p \leq 1$

In this section we give a molecular characterization of the weighted Hardy spaces $H^p_{\mathcal{T}}(w)$ for $0 < p \leq 1$. The results are the following:

Theorem 4.18. *Given $w \in A_\infty$ and $0 < p \leq 1$, let $H^p_L(w)$ be the fixed molecular Hardy space as in Remark 4.12. For every $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $\varepsilon > 0$, and $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$, the following spaces are isomorphic to $H^p_L(w)$ (and therefore one another) with equivalent norms*

$$H^p_{L,q,\varepsilon,M}(w); \qquad H^p_{\mathcal{S}_{m,\mathrm{H}},q}(w),\ m \in \mathbb{N}; \qquad H^p_{\mathrm{G}_{m,\mathrm{H}},q}(w),\ m \in \mathbb{N}_0; \quad \text{and} \qquad H^p_{\mathcal{G}_{m,\mathrm{H}},q}(w),\ m \in \mathbb{N}_0.$$

In particular, none of these spaces depend (modulo isomorphism) on the choice of the allowable parameters q, ε, M, and m.

Theorem 4.19. *Given $w \in A_\infty$ and $0 < p \leq 1$, let $H^p_L(w)$ be the fixed molecular Hardy space as in Remark 4.12. For every $q \in \mathcal{W}_w(p_-(L), p_+(L))$, the following spaces are isomorphic to $H^p_L(w)$ (and therefore one another) with equivalent norms*

$$H^p_{\mathcal{S}_{K,\mathrm{P}},q}(w),\ K \in \mathbb{N}; \qquad H^p_{\mathrm{G}_{K,\mathrm{P}},q}(w),\ K \in \mathbb{N}_0; \qquad \text{and} \qquad H^p_{\mathcal{G}_{K,\mathrm{P}},q}(w),\ K \in \mathbb{N}_0.$$

In particular, none of these spaces depend (modulo isomorphism) on the choice of q, and K.

Theorem 4.20. *Given $w \in A_\infty$ and $0 < p \leq 1$, let $H^p_L(w)$ be the fixed molecular Hardy space as in Remark 4.12. For every $q \in \mathcal{W}_w(p_-(L), p_+(L))$, the following spaces are isomorphic to $H^p_L(w)$ (and therefore one another) with equivalent norms*

$$H^p_{\mathcal{N}_{\mathrm{H}},q}(w) \qquad \text{and} \qquad H^p_{\mathcal{N}_{\mathrm{P}},q}(w).$$

In particular, none of these spaces depend (modulo isomorphism) on the choice of q.

Operators applied to molecules

One of the first steps when proving the above theorems, and in particular, to see that $H^p_L(w) \subset H^p_{\mathcal{T}}(w)$ (where $\mathcal{T}$ is any of the conical square functions in (1.26)-(1.31), or a non-tangential maximal functions in (1.32)), is to study the $L^p(w)$ norm of $\mathcal{T}$ applied to molecules. In fact, we have that $\|\mathcal{T}\mathfrak{m}\|_{L^p(w)} \leq C$, uniformly in $\mathfrak{m}$.

Proposition 4.21. *Let $w \in A_\infty$ and let $\{\mathcal{T}_t\}_{t>0}$ be a family of sublinear operators satisfying the following conditions:*

(*a*) *$\{\mathcal{T}_t\}_{t>0} \in \mathcal{F}_\infty(L^{p_0} \to L^2)$ for all $p_-(L) < p_0 \leq 2$.*

(*b*) *$\widehat{S} f(x) := \left(\iint_{\Gamma(x)} |\mathcal{T}_t f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}$ is bounded on $L^q(w)$ for every $q \in \mathcal{W}_w(p_-(L), p_+(L))$.*

(*c*) *For every* $\lambda > 0$*, there exists* $C_\lambda > 0$ *such that for every* $t > 0$ *it follows that*

$$\mathcal{T}_{\sqrt{1+\lambda}\,t} = C_\lambda \mathcal{T}_t \circ e^{-\lambda t^2 L}.$$

In particular for $\lambda = 1$*, there exists* $C > 0$ *such that for every* $t > 0$ *there holds* $\mathcal{T}_t = C\mathcal{T}_{\frac{t}{\sqrt{2}}} \circ e^{-\frac{t^2}{2}L}$.

Then, for every $\mathfrak{m}$*, a* $(w, q, p, \varepsilon, M)$*-molecule with* $q \in \mathcal{W}_w(p_-(L), p_+(L))$*,* $0 < p \leq 1$*,* $\varepsilon > 0$*, and* $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$*, it follows that* $\|\widehat{S}\,\mathfrak{m}\|_{L^p(w)} \lesssim 1$*, with constants independent of* $\mathfrak{m}$.

Assuming this result momentarily, we obtain the following.

Proposition 4.22. *Let* S *be any of the conical square functions considered in* (1.26)-(1.31). *For every* $w \in A_\infty$ *and* $\mathfrak{m}$ *a* $(w, q, p, \varepsilon, M)$*-molecule with* $q \in \mathcal{W}_w(p_-(L), p_+(L))$*,* $0 < p \leq 1$*,* $\varepsilon > 0$*, and* $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$*, there hold*

(*a*) $\|S\,\mathfrak{m}\|_{L^p(w)} \leq C$.

(*b*) *For all* $f \in \mathbb{H}^p_{L,q,\varepsilon,M}(w)$*,* $\|Sf\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)}$.

Proof. Notice that, in view of Theorems 3.3 and 3.4, and Remark 3.22, and the fact that $\mathcal{S}_{\mathrm{H}} f \leq \frac{1}{2}\mathcal{G}_{\mathrm{H}} f$, to prove part (*a*) it suffices to show the desired estimate for $\mathcal{G}_{\mathrm{H}}$. To this end, we observe that $|t\nabla_{y,t} e^{-t^2L} f|^2 = |t\nabla_y e^{-t^2L} f|^2 + 4|t^2 L e^{-t^2L} f|^2$. Besides, both $\mathcal{T}_t := t\nabla_y e^{-t^2L}$ and $\mathcal{T}_t := t^2 L e^{-t^2L}$ satisfy the hypotheses of Proposition 4.21: (*a*) follows from the off-diagonal estimates that the families $\mathcal{T}_t := t\nabla_y e^{-t^2L}$ and $\mathcal{T}_t := t^2 L e^{-t^2L}$ satisfy (see [3] or Section 1.3.1); (*b*) is contained in Theorem 3.1, part (*a*); and finally (*c*) follows from easy calculations. Thus, we can apply Proposition 4.21 and obtain the desired estimate.

As for part (*b*), fix $w \in A_\infty$ and take $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \leq 1$, $\varepsilon > 0$, and $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$. Then, for $f \in \mathbb{H}^p_{L,q,\varepsilon,M}(w)$, there exists a $(w, q, p, \varepsilon, M)$-representation of f, $f = \sum_{i=1}^\infty \lambda_i \mathfrak{m}_i$, such that

$$\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{\frac{1}{p}} \leq 2\|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)}.$$

On the other hand, since $\sum_{i=1}^\infty \lambda_i \mathfrak{m}_i$ converges in $L^q(w)$ and since for any choice of S, we have that S is a sublinear operator bounded on $L^q(w)$ (see Theorems 3.1 and 3.2, and Remark 3.22). This, part (*a*), and the fact that $0 < p \leq 1$ imply

$$\|Sf\|_{L^p(w)} = \left\|S\left(\sum_{i=1}^\infty \lambda_i \mathfrak{m}_i\right)\right\|_{L^p(w)} \leq \left(\sum_{i=1}^\infty |\lambda_i|^p \|S\,\mathfrak{m}_i\|^p_{L^p(w)}\right)^{\frac{1}{p}} \leq C\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{\frac{1}{p}} \lesssim \|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)},$$

as desired. □

Proof of Proposition 4.21.

Fix $w \in A_\infty$ and $\mathfrak{m}$ a $(w, q, p, \varepsilon, M)$-molecule with $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $\varepsilon > 0$, and $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$, and let Q be a cube associated with $\mathfrak{m}$. Since $w \in A_{\frac{q}{p_-(L)}}$ we can pick $p_-(L) < p_0 < 2$, close enough to $p_-(L)$, so that $w \in A_{\frac{q}{p_0}}$ and simultaneously

$$M > \frac{n}{2}\left(\frac{r_w p_0}{p p_-(L)} - \frac{1}{p_0}\right). \tag{4.23}$$

For every $j, i \geq 1$, consider $Q_i := 2^{i+1}Q$, $\mathfrak{m}_i := \mathfrak{m}\mathbf{1}_{C_i(Q)}$, and $C_{ji} := C_j(Q_i)$. Note that

$$|\mathcal{T}_t\mathfrak{m}(y)| \leq |\mathcal{T}_t\mathfrak{m}(y)|\mathbf{1}_{(0,\ell(Q))}(t) + |\mathcal{T}_t\mathfrak{m}(y)|\mathbf{1}_{[\ell(Q),\infty)}(t) =: F_1(y,t) + F_2(y,t),$$

and therefore, recalling (h) in Notation, since $0 < p \leq 1$,

$$\|\widehat{S}\,\mathfrak{m}\|_{L^p(w)}^p \leq \big\||||F_1|||_{\Gamma(\cdot)}\big\|_{L^p(w)}^p + \big\||||F_2|||_{\Gamma(\cdot)}\big\|_{L^p(w)}^p =: I + II.$$

We estimate each term in turn. Note first that

$$F_1(y,t) \leq \sum_{i\geq 1} |\mathcal{T}_t\mathfrak{m}_i(y)|\mathbf{1}_{(0,\ell(Q))}(t) =: \sum_{i\geq 1} F_{1,i}(y,t).$$

Then,

$$I \lesssim \sum_{i\geq 1} \big\||||F_{1,i}|||_{\Gamma(\cdot)}\big\|_{L^p(16Q_i,w)}^p + \sum_{j\geq 4}\sum_{i\geq 1} \big\||||F_{1,i}|||_{\Gamma(\cdot)}\big\|_{L^p(C_{ji},w)}^p =: \sum_{i\geq 1} I_i + \sum_{j\geq 4}\sum_{i\geq 1} I_{ji}. \tag{4.24}$$

The estimate of I_i follows applying Hölder's inequality, hypothesis (b), (1.11), and (4.8) (for $k = 0$). Then,

$$I_i \leq \|\widehat{S}\,\mathfrak{m}_i\|_{L^p(16Q_i,w)}^p \lesssim w(16Q_i)^{1-\frac{p}{q}}\|\widehat{S}\,\mathfrak{m}_i\|_{L^q(w)}^p \lesssim w(Q_i)^{1-\frac{p}{q}}\|\mathfrak{m}_i\|_{L^q(w)}^p \leq 2^{-ip\varepsilon}. \tag{4.25}$$

To estimate I_{ji}, note that, for every $j \geq 4$ and $i \geq 1$, $0 < t \leq \ell(Q)$, and $x \in C_{ji}$, it follows that $B(x,t) \subset 2^{j+2}Q_i \setminus 2^{j-1}Q_i$. This, hypothesis (a), and Lemma 4.9 imply that

$$\begin{aligned}\left(\int_{B(x,t)} |\mathcal{T}_t\mathfrak{m}_i(y)|^2\,dy\right)^{\frac{1}{2}} &\leq \left(\int_{2^{j+2}Q_i\setminus 2^{j-1}Q_i} |\mathcal{T}_t\mathfrak{m}_i(y)|^2\,dy\right)^{\frac{1}{2}} \\ &\leq t^{-n\left(\frac{1}{p_0}-\frac{1}{2}\right)} e^{-c\frac{4^j\ell(Q_i)^2}{t^2}}\|\mathfrak{m}_i\|_{L^{p_0}(\mathbb{R}^n)} \lesssim t^{-n\left(\frac{1}{p_0}-\frac{1}{2}\right)} e^{-c\frac{4^j\ell(Q_i)^2}{t^2}} 2^{-i\varepsilon} w(Q_i)^{-\frac{1}{p}}|Q_i|^{\frac{1}{p_0}}.\end{aligned}$$

Then, (1.11) and easy calculations lead to

$$\begin{aligned}I_{ji} &\lesssim 2^{-ip\varepsilon}w(Q_i)^{-1}|Q_i|^{\frac{p}{p_0}}\int_{C_{ji}}\left(\int_0^{\ell(Q)} t^{-2n\left(\frac{1}{p_0}-\frac{1}{2}\right)}e^{-c\frac{4^j\ell(Q_i)^2}{t^2}}\frac{dt}{t^{n+1}}\right)^{\frac{p}{2}} w(x)dx \\ &\lesssim 2^{-ip\varepsilon}w(Q_i)^{-1}|Q_i|^{\frac{p}{p_0}} w(2^{j+1}Q_i)\big(4^j\,\ell(Q_i)^2\big)^{-\frac{np}{2p_0}}\left(\int_{2^{j+i}}^{\infty} s^{\frac{2n}{p_0}}e^{-cs^2}\frac{ds}{s}\right)^{\frac{p}{2}} \lesssim 2^{-ip\varepsilon}e^{-c4^j}.\end{aligned}$$

Plugging this and (4.25) into (4.24), we finally conclude the desired estimate for I:

$$I \lesssim \sum_{i\geq 1} 2^{-ip\varepsilon} + \sum_{j\geq 4}\sum_{i\geq 1} 2^{-ip\varepsilon}e^{-c4^j} \lesssim 1. \tag{4.26}$$

We turn now to estimate II. First, set

$$B_Q := \left(I - e^{-\ell(Q)^2L}\right)^M \qquad \text{and} \qquad A_Q := I - B_Q,$$

and observe that

$$F_2(y,t) \leq |\mathcal{T}_tA_Q\mathfrak{m}(y)|\mathbf{1}_{[\ell(Q),\infty)}(t) + |\mathcal{T}_tB_Q\mathfrak{m}(y)|\mathbf{1}_{[\ell(Q),\infty)}(t) =: F_3(y,t) + F_4(y,t). \tag{4.27}$$

We start estimating the term related to F_3. To do that, consider

$$h(y) := \sum_{i\geq 1} h_i(y) := \sum_{i\geq 1} (\ell(Q)^2L)^{-M}\mathfrak{m}(y)\,\mathbf{1}_{C_i(Q)}(y),$$

and note that

$$F_3(y,t) \le \sum_{i\ge 1} |\mathcal{T}_t A_Q(\ell(Q)^2 L)^M h_i(y)| \mathbf{1}_{[\ell(Q),\infty)}(t).$$

Then, we obtain

$$\begin{aligned}\Big\| |||F_3|||_{\Gamma(\cdot)} \Big\|^p_{L^p(w)} &\le \sum_{i\ge 1} \left\| \left(\iint_{\Gamma(\cdot)} |\mathcal{T}_t A_Q(\ell(Q)^2 L)^M h_i(y)|^2 \mathbf{1}_{[\ell(Q),\infty)} \frac{dy\,dt}{t^{n+1}} \right)^{\frac12} \right\|^p_{L^p(16Q_i,w)} \\ &\quad + \sum_{j\ge 4}\sum_{i\ge 1} \left\| \left(\iint_{\Gamma(\cdot)} |\mathcal{T}_t A_Q(\ell(Q)^2 L)^M h_i(y)|^2 \mathbf{1}_{[\ell(Q),\infty)} \frac{dy\,dt}{t^{n+1}} \right)^{\frac12} \right\|^p_{L^p(C_{ji},w)} \\ &=: \sum_{i\ge 1} II_i + \sum_{j\ge 4}\sum_{i\ge 1} II_{ji}.\end{aligned}$$

Before estimating II_i and II_{ji}, note that by Proposition 1.41, one can easily obtain that $A_Q(\ell(Q)^2L)^M$ is bounded on $L^q(w)$ uniformly on Q since $q \in \mathcal{W}_w(p_-(L), p_+(L))$ and

$$A_Q(\ell(Q)^2 L)^M = (I-(I-e^{-\ell(Q)^2 L})^M)(\ell(Q)^2L)^M = \sum_{k=1}^M C_{k,M} (k\ell(Q)^2 L)^M e^{-k\ell(Q)^2 L}.$$

This, Hölder's inequality, hypothesis (b), (1.11), and (4.8) imply

$$II_i \le w(16Q_i)^{1-\frac{p}{q}} \|A_Q(\ell(Q)^2L)^M h_i\|^p_{L^q(w)} \lesssim w(Q_i)^{1-\frac{p}{q}} \|h_i\|^p_{L^q(w)} \lesssim 2^{-ip\varepsilon}. \tag{4.28}$$

We turn now to estimate II_{ji}. Note that for every $x \in C_{ji}$, $j \ge 4$, $i \ge 1$

$$\big\{(y,t) : y \in B(x,t),\ t \ge \ell(Q)\big\} \subset E_1 \cup E_2 \cup E_3,$$

where

$$E_1 := \left(2^{j+2}Q_i \setminus 2^{j-1}Q_i\right) \times \left[\ell(Q), 2^{j-2}\ell(Q_i)\right], \qquad E_2 := 2^j Q_i \times \left(2^{j-2}\ell(Q_i),\infty\right),$$

and

$$E_3 := \Big(\bigcup_{l\ge j} C_l(Q_i)\Big) \times \Big(2^{j-2}\ell(Q_i),\infty\Big).$$

Consequently,

$$II_{ji} \le w(2^{j+1}Q_i) \sum_{l=1}^3 \left(\iint_{E_l} |\mathcal{T}_t A_Q(\ell(Q)^2L)^M h_i(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p}{2}} =: w(2^{j+1}Q_i)\sum_{l=1}^3 G_l.$$

Now observe that hypothesis (c) with $\lambda = 1$ implies

$$|\mathcal{T}_t A_Q(\ell(Q)^2L)^M h_i(y)| = C\,|\mathcal{T}_{\frac{t}{\sqrt2}} e^{-\frac{t^2}{2}L} A_Q(\ell(Q)^2L)^M h_i(y)|.$$

Besides,

$$e^{-\frac{t^2}{2}L} A_Q(\ell(Q)^2L)^M = \sum_{k=1}^M C_{k,M} \left(\frac{\ell(Q)^2}{s_{Q,t}^2}\right)^M \left(s_{Q,t}^2 L\right)^M e^{-s_{Q,t}^2 L}, \quad \text{where} \quad s_{Q,t} := \left(k\ell(Q)^2 + \frac{t^2}{2}\right)^{\frac12}.$$

Then, applying hypothesis (a), the fact that $\{(t^2L)^M e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^{p_0} \to L^{p_0})$ together with Lemma 1.35 (see also [54, Lemma 2.3]), and Lemma 4.9, we have

$$G_1 \lesssim \sum_{k=1}^M \left(\int_{\ell(Q)}^{2^{j-2}\ell(Q_i)} \left(\frac{\ell(Q)^2}{s_{Q,t}^2}\right)^{2M} \int_{2^{j+2}Q_i\setminus 2^{j-1}Q_i} |\mathcal{T}_{\frac{t}{\sqrt2}} \left(s_{Q,t}^2L\right)^M e^{-s_{Q,t}^2L} h_i(y)|^2 dy \frac{dt}{t^{n+1}} \right)^{\frac{p}{2}}$$

$$\lesssim \left(\int_{\ell(Q)}^{2^{j-2}\ell(Q_i)} \ell(Q)^{4M} t^{-4M-\frac{2n}{p_0}} e^{-c\frac{4^{j+i}\ell(Q)^2}{t^2}} \frac{dt}{t}\right)^{\frac{p}{2}} 2^{-ip\varepsilon} w(Q_i)^{-1} |Q_i|^{\frac{p}{p_0}}$$
$$\lesssim 2^{-jp\left(2M+\frac{n}{p_0}\right)} 2^{-ip(2M+\varepsilon)} w(Q_i)^{-1}.$$

Similarly,

$$G_2 \lesssim \left(\int_{2^{j-2}\ell(Q_i)}^{\infty} \ell(Q)^{4M} t^{-4M-\frac{2n}{p_0}} \frac{dt}{t}\right)^{\frac{p}{2}} 2^{-ip\varepsilon} w(Q_i)^{-1} |Q_i|^{\frac{p}{p_0}} \lesssim 2^{-jp\left(2M+\frac{n}{p_0}\right)} 2^{-ip(2M+\varepsilon)} w(Q_i)^{-1},$$

and

$$G_3 \lesssim \sum_{l\geq j} \left(\int_0^{\infty} s^{4M+\frac{2n}{p_0}} e^{-cs^2} \frac{ds}{s}\right)^{\frac{p}{2}} (2^{(l+i)}\ell(Q))^{-p\left(2M+\frac{n}{p_0}\right)} \ell(Q)^{2pM} 2^{-ip\varepsilon} w(Q_i)^{-1} |Q_i|^{\frac{p}{p_0}}$$
$$\lesssim 2^{-jp\left(2M+\frac{n}{p_0}\right)} 2^{-ip(2M+\varepsilon)} w(Q_i)^{-1}.$$

Collecting the estimates for G_1, G_2, and G_3 gives us

$$II_{ji} \lesssim \frac{w(2^{j+1}Q_i)}{w(Q_i)} 2^{-jp\left(2M+\frac{n}{p_0}\right)} 2^{-ip(2M+\varepsilon)} \lesssim 2^{-j\left(2pM+\frac{np}{p_0}-\frac{r_w p_0 n}{p_-(L)}\right)} 2^{-ip(2M+\varepsilon)},$$

where we have used that $w \in A_{\frac{r_w p_0}{p_-(L)}}$, by the definition of r_w and the fact that $p_-(L) < p_0$, and (1.11). Gathering this and (4.28), conclude that (4.23) yields

$$\left\| |||F_3|||_{\Gamma(\cdot)} \right\|_{L^p(w)}^p \lesssim \sum_{i\geq 1} 2^{-ip\varepsilon} + \sum_{j\geq 4}\sum_{i\geq 1} 2^{-j\left(2pM+\frac{np}{p_0}-\frac{r_w p_0 n}{p_-(L)}\right)} 2^{-ip(2M+\varepsilon)} \lesssim 1. \tag{4.29}$$

We next estimate F_4:

$$\begin{aligned}\left\| |||F_4|||_{\Gamma(\cdot)} \right\|_{L^p(w)}^p &\leq \sum_{i\geq 1} \left\| \left(\iint_{\Gamma(\cdot)} |\mathcal{T}_t B_Q \mathfrak{m}_i(y)|^2 \mathbf{1}_{[\ell(Q),\infty)}(t) \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}} \right\|_{L^p(16Q_i,w)}^p \\ &\quad + \sum_{i\geq 1}\sum_{j\geq 4} \left\| \left(\iint_{\Gamma(\cdot)} |\mathcal{T}_t B_Q \mathfrak{m}_i(y)|^2 \mathbf{1}_{[\ell(Q),\infty)}(t) \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}} \right\|_{L^p(C_{ji},w)}^p \\ &=: \sum_{i\geq 1} III_i + \sum_{i\geq 1}\sum_{j\geq 4} III_{ji}.\end{aligned}$$

Note that the fact that the semigroup $\{e^{-tL}\}_{t>0}$ is uniformly bounded on $L^q(w)$, since it was assumed that $q \in \mathcal{W}_w(p_-(L), p_+(L)) \subset \widetilde{\mathcal{J}}_w(L)$ (see Proposition 1.41, part (a)), easily gives that B_Q is bounded on $L^p(w)$ uniformly in Q. Hence, Hölder's inequality, hypothesis (b), and (4.8) (for $k = 0$), yield

$$III_i \lesssim w(16Q_i)^{1-\frac{p}{q}} \|\widehat{S} B_Q \mathfrak{m}_i\|_{L^q(w)}^p \lesssim w(16Q_i)^{1-\frac{p}{q}} \|\mathfrak{m}_i\|_{L^q(w)}^p \lesssim 2^{-ip\varepsilon}. \tag{4.30}$$

Now, change the variable t into $\sqrt{1+M}t$ and use hypothesis (c) to obtain

$$\begin{aligned}III_{ji} &\lesssim \left\| \left(\iint_{\Gamma^{\sqrt{1+M}}(\cdot)} |\mathcal{T}_{\sqrt{1+M}t} B_Q \mathfrak{m}_i(y)|^2 \mathbf{1}_{[\ell(Q)/\sqrt{1+M},\infty)}(t) \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}} \right\|_{L^p(C_{ji},w)}^p \\ &\approx \left\| \left(\iint_{\Gamma^{\sqrt{1+M}}(\cdot)} |\mathcal{T}_t e^{-Mt^2 L} B_Q \mathfrak{m}_i(y)|^2 \mathbf{1}_{[\ell(Q)/\sqrt{1+M},\infty)}(t) \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}} \right\|_{L^p(C_{ji},w)}^p.\end{aligned}$$

Setting $\theta_M = (1+M)^{-\frac{1}{2}}$, for every $x \in C_{ji}$, we have

$$\left\{(y,t) : y \in B(x, \theta_M^{-1} t),\ \theta_M \ell(Q) < t < \infty\right\} \subset \widetilde{E}_1 \cup \widetilde{E}_2 \cup \widetilde{E}_3,$$

where

$$\widetilde{E}_1 := \left(2^{j+2}Q_i \setminus 2^{j-1}Q_i\right) \times \left(\theta_M \ell(Q), 2^{j-2}\theta_M \ell(Q_i)\right], \qquad \widetilde{E}_2 := 2^j Q_i \times \left(2^{j-2}\theta_M \ell(Q_i), \infty\right),$$

and

$$\widetilde{E}_3 := \left(\bigcup_{l \ge j} C_l(Q_i)\right) \times \left(2^{j-2}\theta_M \ell(Q_i), \infty\right).$$

Then we have that

$$III_{ji} \lesssim w(2^{j+1}Q_i) \sum_{l=1}^{3} \left(\iint_{\widetilde{E}_l} |\mathcal{T}_t e^{-Mt^2 L} B_Q \mathfrak{m}_i(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p}{2}} =: w(2^{j+1}Q_i) \sum_{l=1}^{3} \widetilde{G}_l.$$

At this point we proceed much as in the estimates for G_1, G_2, and G_3. Applying Proposition 1.37 with $s = \ell(Q)$, and $p = q = p_0$, we have that

$$III_{ji} \lesssim \frac{w(2^{j+1}Q_i)}{w(Q_i)} 2^{-jp\left(2M + \frac{n}{p_0}\right)} 2^{-ip(2M+\varepsilon)} \lesssim 2^{-j\left(2pM + \frac{np}{p_0} - \frac{r_w p_0 n}{p_-(L)}\right)} 2^{-ip(2M+\varepsilon)},$$

where we have used that $w \in A_{\frac{r_w p_0}{p_-(L)}}$, by the definition of r_w and the fact that $p_-(L) < p_0$, and (1.11). From this and (4.30), we conclude that (4.23) yields

$$\left\| |||F_4|||_{\Gamma(\cdot)} \right\|^p_{L^p(w)} \lesssim \sum_{i \ge 1} 2^{-ip\varepsilon} + \sum_{i \ge 1} \sum_{j \ge 4} 2^{-j\left(2pM + \frac{np}{p_0} - \frac{r_w p_0 n}{p_-(L)}\right)} 2^{-ip(2M+\varepsilon)} \lesssim 1.$$

Using, this, (4.29), and (4.27), we conclude that $II \lesssim 1$, which, together with (4.26), gives the desired estimate: $\|\widehat{S}\,\mathfrak{m}\|_{L^p(w)} \lesssim 1$. □

Similarly, for the non-tangential maximal functions considered in (1.32), we have:

Proposition 4.31. *Let $w \in A_\infty$, let $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \le 1$, $\varepsilon > 0$, and $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$, and let $\mathfrak{m}$ be a $(w, q, p, \varepsilon, M)$-molecule. There hold*

(*a*) $\|\mathcal{N}_{\mathrm{H}}\mathfrak{m}\|_{L^p(w)} + \|\mathcal{N}_{\mathrm{P}}\mathfrak{m}\|_{L^p(w)} \le C$.

(*b*) *For all $f \in \mathbb{H}^p_{L,q,\varepsilon,M}(w)$,*

$$\|\mathcal{N}_{\mathrm{H}} f\|_{L^p(w)} + \|\mathcal{N}_{\mathrm{P}} f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)}.$$

Proof. Assuming part (*a*), the proof of part (*b*) is similar to that of Proposition 4.22, part (*b*), but applying Theorem 3.25 instead of Theorems 3.1 and 3.2.

Let us prove part (*a*). Fix $w \in A_\infty$, $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \le 1$, $\varepsilon > 0$, $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$. Then, take $\mathfrak{m}$ a $(w, q, p, \varepsilon, M)$-molecule, and Q a cube associated with $\mathfrak{m}$. Besides we fix p_0, q_0, and $\widehat{r}$ with $p_-(L) < p_0 < \min\{2, q\} \le \max\{2, q\} < q_0 < p_+(L)$ and $\widehat{r} > r_w$ so that $w \in A_{\frac{q}{p_0}} \cap RH_{\left(\frac{q_0}{q}\right)'}$ and $M > \frac{n}{2}\left(\frac{\widehat{r}}{p} - \frac{1}{p_0}\right)$.

We start by dealing with $\mathcal{N}_{\mathrm{H}}$. For every $x \in \mathbb{R}^n$, we have

$$\mathcal{N}_{\mathrm{H}}\mathfrak{m}(x) \le \left(\sup_{(y,t) \in \Gamma(x),\, 0 < t \le \ell(Q)} \int_{B(y,t)} |e^{-t^2 L}\mathfrak{m}(z)|^2 \frac{dz}{t^n} \right)^{\frac{1}{2}}$$

$$+\left(\sup_{(y,t)\in\Gamma(x),\, t>\ell(Q)}\int_{B(y,t)}|e^{-t^2L}\mathfrak{m}(z)|^2\frac{dz}{t^n}\right)^{\frac12} =: F_1\mathfrak{m}(x)+F_2\mathfrak{m}(x).$$

Besides, recalling the notation introduced in (4.6), we can write $\mathfrak{m}=\sum_{i\geq1}\mathfrak{m}\mathbf{1}_{C_i(Q)} =: \sum_{i\geq1}\mathfrak{m}_i$. Hence, since $0<p\leq 1$,

$$\|F_1\mathfrak{m}\|_{L^p(w)}^p \lesssim \sum_{i\geq1}\|\mathbf{1}_{16Q_i}F_1\mathfrak{m}_i\|_{L^p(w)}^p+\sum_{i\geq1}\sum_{j\geq4}\|\mathbf{1}_{C_j(Q_i)}F_1\mathfrak{m}_i\|_{L^p(w)}^p =: \sum_{i\geq1}I_i+\sum_{i\geq1}\sum_{j\geq4}I_{ji}. \tag{4.32}$$

To estimate I_i, we apply Hölder's inequality, Theorem 3.25, and (4.8) for $k=0$. Then,

$$I_i\lesssim w(Q_i)^{1-\frac{p}{q}}\|\mathcal{N}_{\mathrm{H}}\mathfrak{m}_i\|_{L^q(w)}^p\lesssim w(Q_i)^{1-\frac{p}{q}}\|\mathfrak{m}_i\|_{L^q(w)}^p\leq 2^{-ip\varepsilon}. \tag{4.33}$$

As for I_{ji}, note that for every $x\in C_j(Q_i)$, $0<t\leq\ell(Q)$, and $(y,t)\in\Gamma(x)$, we have that $B(y,t)\subset 2^{j+2}Q_i\setminus 2^{j-1}Q_i$. Then, applying that $\{e^{-t^2L}\}_{t>0}\in\mathcal{F}_\infty(L^{p_0}\to L^2)$ and Lemma 4.9, we get

$$\begin{aligned}F_1\mathfrak{m}_i(x)&\leq\left(\sup_{0<t\leq\ell(Q)}\int_{2^{j+1}Q_i\setminus 2^{j-1}Q_i}|e^{-t^2L}\mathfrak{m}_i(z)|^2\frac{dz}{t^n}\right)^{\frac12}\leq\sup_{0<t\leq\ell(Q)}t^{-\frac{n}{p_0}}e^{-c\frac{4^{j+i}\ell(Q)^2}{t^2}}\|\mathfrak{m}_i\|_{L^{p_0}(\mathbb{R}^n)}\\&\lesssim w(Q_i)^{-\frac1p}|Q_i|^{\frac{1}{p_0}}2^{-i\varepsilon}e^{-c4^{j+i}}(2^{j+i}\ell(Q))^{-\frac{n}{p_0}}.\end{aligned}$$

Therefore, taking the norm in $L^p(w)$ in the previous expression and using that $w\in A_\infty$, we obtain that $I_{ji}\lesssim e^{-c4^{j+i}}$. This, (4.32), and (4.33) yield $\|F_1\mathfrak{m}\|_{L^p(w)}\leq C$.

We turn now to estimate the norm in $L^p(w)$ of $F_2\mathfrak{m}$. Considering $B_Q:=(I-e^{-\ell(Q)^2L})^M$, $A_Q:=I-B_Q$, and $\widetilde{\mathfrak{m}}:=(\ell(Q)^2L)^{-M}\mathfrak{m}$, and noticing that we can write $\widetilde{\mathfrak{m}}=\sum_{i\geq1}\widetilde{\mathfrak{m}}\mathbf{1}_{C_i(Q)} =: \sum_{i\geq1}\widetilde{\mathfrak{m}}_i$. Then, for every $x\in\mathbb{R}^n$,

$$\mathfrak{m}(x)=B_Q\mathfrak{m}(x)+A_Q\mathfrak{m}(x)=\sum_{i\geq1}\left(B_Q\mathfrak{m}_i(x)+\sum_{k=1}^M C_{k,M}(k\ell(Q)^2L)^Me^{-k\ell(Q)^2L}\widetilde{\mathfrak{m}}_i(x)\right).$$

Besides, proceeding as in (4.33) and applying the fact that, for every $1\leq k\leq M$, $(k\ell(Q)^2L)^Me^{-k\ell(Q)^2L}$ and B_Q are bounded on $L^q(w)$ (see Proposition 1.41), we have that

$$\sum_{i\geq1}\left(\left\|\mathbf{1}_{16Q_i}F_2B_Q\mathfrak{m}_i\right\|_{L^p(w)}^p+\sum_{k=1}^M C_{k,M}\left\|\mathbf{1}_{16Q_i}F_2(k\ell(Q)^2L)^Me^{-k\ell(Q)^2L}\widetilde{\mathfrak{m}}_i\right\|_{L^p(w)}^p\right)\leq C. \tag{4.34}$$

Next, consider $\theta_M:=\sqrt{M+1}$ and note that, for every $j\geq4$, $i\geq1$, $x\in C_j(Q_i)$, $\ell(Q)/\theta_M<t\leq 2^{j-3}\ell(Q_i)/\theta_M$, and $(y,\theta_Mt)\in\Gamma(x)$, we have that $B(y,\theta_Mt)\subset 2^{j+2}Q_i\setminus 2^{j-1}Q_i$. Therefore, since $\{e^{-t^2L}\}_{t>0}\in\mathcal{F}_\infty(L^{p_0}-L^2)$ and by the $L^{p_0}(\mathbb{R}^n)-L^{p_0}(\mathbb{R}^n)$ off-diagonal estimates satisfied by $\{(e^{-t^2L}-e^{-(t^2+\ell(Q)^2)L})^M\}_{t>0}$ (see (1.38)), applying Lemma 1.35 (see also [54, Lemma 2.3]), and Lemma 4.9, we have

$$\begin{aligned}F_2B_Q\mathfrak{m}_i(x)&\lesssim\|\mathfrak{m}_i\|_{L^{p_0}(\mathbb{R}^n)}\left(\sup_{\frac{\ell(Q)}{\theta_M}<t\leq\frac{2^{j-3}\ell(Q_i)}{\theta_M}}\left(\frac{\ell(Q)}{t}\right)^{2M}t^{-\frac{n}{p_0}}e^{-c\frac{4^{j+i}\ell(Q)^2}{t^2}}+\sup_{\frac{2^{j-3}\ell(Q_i)}{\theta_M}<t}\left(\frac{\ell(Q)}{t}\right)^{2M}t^{-\frac{n}{p_0}}\right)\\&\lesssim w(Q_i)^{-\frac1p}\,2^{-i(2M+\varepsilon)}2^{-j\left(2M+\frac{n}{p_0}\right)}.\end{aligned}$$

Then, taking the norm in $L^p(w)$ in the previous inequality and using (1.11), we obtain that

$$\|\mathbf{1}_{C_j(Q_i)}F_2B_Q\mathfrak{m}_i\|_{L^p(w)}\lesssim 2^{-i(2M+\varepsilon)}2^{-j\left(2M+\frac{n}{p_0}-\frac{\widehat{r}n}{p}\right)}, \tag{4.35}$$

for all $j \geq 4$ and $i \geq 1$.

Note now that, for every $j \geq 4$, $i \geq 1$, $x \in C_j(Q_i)$, $\ell(Q)/\sqrt{2} < t \leq 2^{j-3}\ell(Q_i)/\sqrt{2}$, and $(y, \sqrt{2}t) \in \Gamma(x)$, we have that $B(y, \sqrt{2}t) \subset 2^{j+2}Q_i \setminus 2^{j-1}Q_i$. Then, proceeding as in the estimate of $F_2 B_Q \mathfrak{m}_i$, but using this time the off-diagonal estimates satisfied by the family $\{t^2 L e^{-t^2 L}\}_{t>0}$ instead of the ones satisfied by $\{(e^{-t^2 L} - e^{-(t^2+\ell(Q)^2)L})^M\}_{t>0}$, we have that, for every $1 \leq k \leq M$,

$$
\begin{aligned}
&F_2\left((k\ell(Q)^2 L)^M e^{-k\ell(Q)^2 L}\widetilde{\mathfrak{m}}_i\right)(x)\\
&\lesssim \sup_{(y,\sqrt{2}t)\in\Gamma(x),t>\frac{\ell(Q)}{\sqrt{2}}} \left(\frac{\ell(Q)^2}{t^2+k\ell(Q)^2}\right)^M \left(\int_{B(y,\sqrt{2}t)} \left|e^{-t^2L}((t^2+k\ell(Q)^2)L)^M e^{-(t^2+k\ell(Q)^2)L}\widetilde{\mathfrak{m}}_i(z)\right|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}}\\
&\lesssim \|\widetilde{\mathfrak{m}}_i\|_{L^{p_0}(\mathbb{R}^n)} \left(\sup_{\frac{\ell(Q)}{\sqrt{2}}<t\leq\frac{2^{j-3}\ell(Q_i)}{\sqrt{2}}} \left(\frac{\ell(Q)}{t}\right)^{2M} t^{-\frac{n}{p_0}} e^{-c\frac{4^{j+i}\ell(Q)^2}{t^2}} + \sup_{\frac{2^{j-3}\ell(Q_i)}{\sqrt{2}}<t} \left(\frac{\ell(Q)}{t}\right)^{2M} t^{-\frac{n}{p_0}}\right)\\
&\lesssim w(Q_i)^{-\frac{1}{p}}\, 2^{-i(2M+\varepsilon)} 2^{-j\left(2M+\frac{n}{p_0}\right)}.
\end{aligned}
$$

Then, $\|\mathbf{1}_{C_j(Q_i)} F_2 A_Q \mathfrak{m}\|_{L^p(w)} \lesssim 2^{-i(2M+\varepsilon)} 2^{-j\left(2M+\frac{n}{p_0}-\frac{\widehat{r}n}{p}\right)}$, for all $j \geq 4$ and $i \geq 1$. This, (4.35), and (4.34), and splitting the norm of $F_2\mathfrak{m}$ as in (4.32), allow us to conclude that $\|F_2\mathfrak{m}\|_{L^p(w)} \leq C$.

We consider now $\mathcal{N}_{\mathrm{P}}$. Note that, in the proof of Theorem 3.25, part (*b*) (and following the notation introduced there with $f = \mathfrak{m}$) we saw that $\mathcal{N}_{\mathrm{P}}\mathfrak{m}(x) \lesssim \mathfrak{m}_{\mathrm{P}}\mathfrak{m}(x) + \mathcal{N}_{\mathrm{H}}\mathfrak{m}(x)$. Then, since we have already proved that $\|\mathcal{N}_{\mathrm{H}}\mathfrak{m}\|_{L^p(w)} \leq C$, we just need to consider $\mathfrak{m}_{\mathrm{P}}\mathfrak{m}$. Applying, the subordination formula (1.36), we have that

$$
\begin{aligned}
\mathfrak{m}_{\mathrm{P}}\mathfrak{m}(x) \lesssim& \sup_{(y,t)\in\Gamma(x)} \int_0^{\frac{1}{4}} u^{\frac{1}{2}} \left(\int_{B(y,t)} |(e^{-\frac{t^2}{4u}L} - e^{-t^2L})\mathfrak{m}(z)|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}} \frac{du}{u}\\
&+ \sup_{(y,t)\in\Gamma(x)} \int_{\frac{1}{4}}^{\infty} e^{-u} u^{\frac{1}{2}} \left(\int_{B(y,t)} |(e^{-\frac{t^2}{4u}L} - e^{-t^2L})\mathfrak{m}(z)|^2 \frac{dz}{t^n}\right)^{\frac{1}{2}} \frac{du}{u} =: I + II.
\end{aligned}
$$

Note that II is bounded by the term II (with $f = \mathfrak{m}$) in the proof of Theorem 3.25, part (*b*). Hence

$$
II \lesssim \int_{\frac{1}{4}}^{\infty} e^{-u} \mathcal{S}_{\mathrm{H}}^{4\sqrt{u}} f(x) du.
$$

Next, note that for all $w_0 \in A_\infty$, and for $r_0 > r_w$, applying Minkowski's integral inequality and Proposition 2.43, part (*a*), we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \left(\int_{\frac{1}{4}}^{\infty} e^{-u}\, \mathcal{S}_{\mathrm{H}}^{4\sqrt{u}}\mathfrak{m}(x) du\right)^2 w_0(x)dx &\leq \left(\int_{\frac{1}{4}}^{\infty} e^{-u} \left(\int_{\mathbb{R}^n} (\mathcal{S}_{\mathrm{H}}^{4\sqrt{u}}\mathfrak{m}(x))^2 w_0(x)dx\right)^{\frac{1}{2}} du\right)^2\\
&\leq \left(\int_{\frac{1}{4}}^{\infty} e^{-u}(4\sqrt{u})^{\frac{nr_0}{2}} \left(\int_{\mathbb{R}^n} (\mathcal{S}_{\mathrm{H}}\mathfrak{m}(x))^2 w_0(x)dx\right)^{\frac{1}{2}} du\right)^2\\
&\lesssim \int_{\mathbb{R}^n} (\mathcal{S}_{\mathrm{H}}\mathfrak{m}(x))^2 w_0(x)dx.
\end{aligned}
$$

Hence, applying Theorem 1.46, part (*e*), we obtain for all $0 < \widetilde{r} < \infty$ and $\widetilde{w} \in A_\infty$

$$
\int_{\mathbb{R}^n} \left(\int_{\frac{1}{4}}^{\infty} e^{-u}\, \mathcal{S}_{\mathrm{H}}^{4\sqrt{u}}\mathfrak{m}(x) du\right)^{\widetilde{r}} \widetilde{w}(x)dx \lesssim \int_{\mathbb{R}^n} (\mathcal{S}_{\mathrm{H}}\mathfrak{m}(x))^{\widetilde{r}} \widetilde{w}(x)dx.
$$

In particular, for every $0 < p \leq 1$ and $w \in A_{\frac{q}{p_-(L)}} \cap RH_{\left(\frac{p_+(L)}{q}\right)'}$, we get

$$\int_{\mathbb{R}^n} \left(\int_{\frac{1}{4}}^{\infty} e^{-u}\, \mathcal{S}_{\mathrm{H}}^{4\sqrt{u}} \mathbf{m}(x) du \right)^p w(x)dx \lesssim \int_{\mathbb{R}^n} (\mathcal{S}_{\mathrm{H}} \mathbf{m}(x))^p w(x)dx.$$

Using this and Proposition 4.22, part (a), we conclude that $II \leq C$.

Let us now estimate I. We shall use the notation introduced before for $\mathbf{m}_i$, $\widetilde{\mathbf{m}}_i$, B_Q, and A_Q, and also in (4.6). Proceeding as in the estimate of the term I (with $f = \mathbf{m}$) in the proof of Theorem 3.25, part (b), we have

$$\begin{aligned} I &\lesssim \sum_{l\geq 1} e^{-c4^l} \sup_{(y,t)\in\Gamma(x)} \left(\fint_{B(y,2^{l+1}t)} \left(\int_{\frac{t}{\sqrt{2}}}^{\infty} |r^2 L e^{-r^2L} \mathbf{m}(z)|^2 \frac{dr}{r} \right)^{\frac{p_0}{2}} dz \right)^{\frac{1}{p_0}} \qquad (4.36) \\ &\lesssim \sum_{l\geq 1} e^{-c4^l} \sup_{(y,t)\in\Gamma(x), 0<t\leq\ell(Q)} \left(\fint_{B(y,2^{l+1}t)} \left(\int_0^{\infty} |r^2 L e^{-r^2L} \mathbf{m}(z)|^2 \frac{dr}{r} \right)^{\frac{p_0}{2}} dz \right)^{\frac{1}{p_0}} \\ &\quad + \sum_{l\geq 1} e^{-c4^l} \sup_{(y,t)\in\Gamma(x), t>\ell(Q)} \left(\fint_{B(y,2^{l+1}t)} \left(\int_{\frac{t}{\sqrt{2}}}^{\infty} |r^2 L e^{-r^2L} \mathbf{m}(z)|^2 \frac{dr}{r} \right)^{\frac{p_0}{2}} dz \right)^{\frac{1}{p_0}} \\ &=: \sum_{l\geq 1} e^{-c4^l} \left(F_{1,l}\mathbf{m}(x) + F_{2,l}\mathbf{m}(x) \right). \end{aligned}$$

We first estimate $F_{1,l}\mathbf{m}(x)$. Note that considering the following vertical square functions

$$g_{\mathrm{H},1}\mathbf{m}(x) := \left(\int_0^{\ell(Q)} |r^2 L e^{-r^2L} \mathbf{m}(x)|^2 \frac{dr}{r} \right)^{\frac{1}{2}}$$

and

$$g_{\mathrm{H},2}\widetilde{\mathbf{m}}(x) := \left(\int_{\ell(Q)}^{\infty} \left(\frac{\ell(Q)^2}{r^2} \right)^{2M} |(r^2L)^{M+1} e^{-r^2L} \widetilde{\mathbf{m}}(x)|^2 \frac{dr}{r} \right)^{\frac{1}{2}}.$$

We have that

$$\begin{aligned} F_{1,l}\mathbf{m}(x) &\lesssim \sup_{(y,t)\in\Gamma(x), 0<t\leq\ell(Q)} \left(\fint_{B(y,2^{l+1}t)} |g_{\mathrm{H},1}\mathbf{m}(z)|^{p_0} dz \right)^{\frac{1}{p_0}} \\ &\quad + \sup_{(y,t)\in\Gamma(x), 0<t\leq\ell(Q)} \left(\fint_{B(y,2^{l+1}t)} |g_{\mathrm{H},2}\widetilde{\mathbf{m}}(z)|^{p_0} dz \right)^{\frac{1}{p_0}} =: F_{1,l}^1\mathbf{m}(x) + F_{1,l}^2\widetilde{\mathbf{m}}(x). \qquad (4.37) \end{aligned}$$

Applying Hölder's inequality, by the boundedness on $L^q(w)$ of the maximal operator $\mathcal{M}_{p_0}$ (recall that $w \in A_{\frac{q}{p_0}}$) and the vertical square function $g_{\mathrm{H},1}$ (see [11]), by (4.8), and (1.11), we have that

$$\left\| \mathbf{1}_{2^{l+3}Q_i} F_{1,l}^1 \mathbf{m}_i \right\|_{L^p(w)} \lesssim \|\mathbf{1}_{2^{l+3}Q_i} \mathcal{M}_{p_0}(g_{\mathrm{H},1}\mathbf{m}_i)\|_{L^p(w)} \lesssim w(2^l Q_i)^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{M}_{p_0}(g_{\mathrm{H},1}\mathbf{m}_i)\|_{L^q(w)} \lesssim 2^{l\widehat{r}} 2^{-i\varepsilon}.$$

Now observe that for every $i \geq 1$, $j \geq l+3$, $x \in C_j(Q_i)$, $0 < t \leq \ell(Q)$, and $(y,t) \in \Gamma(x)$ we have that $B(y, 2^{l+1}t) \subset 2^{j+2}Q_i \setminus 2^{j-1}Q_i$. Then, applying Hölder's inequality, the fact that $w \in RH_{\left(\frac{q_0}{q}\right)'}$, Minkowski's integral inequality, the fact that $\{r^2 L e^{-r^2L}\}_{r>0} \in \mathcal{F}(L^{p_0} - L^{q_0})$, and Lemma 4.9, we obtain that

$$\left\| \mathbf{1}_{C_j(Q_i)} F_{1,l}^1 \mathbf{m}_i \right\|_{L^p(w)} \lesssim w(2^{j+1}Q_i)^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{M}_{p_0}(\mathbf{1}_{2^{j+2}Q_i \setminus 2^{j-1}Q_i} g_{\mathrm{H},1}\mathbf{m}_i)\|_{L^q(w)}$$

$$
\begin{aligned}
&\lesssim w(2^{j+1}Q_i)^{\frac{1}{p}-\frac{1}{q}} \|\mathbf{1}_{2^{j+2}Q_i\setminus 2^{j-1}Q_i} g_{\mathrm{H},1}\,\mathfrak{m}_i\|_{L^q(w)}\\
&\lesssim w(2^{j+1}Q_i)^{\frac{1}{p}} |2^{j+1}Q_i|^{-\frac{1}{q_0}} \|\mathbf{1}_{2^{j+2}Q_i\setminus 2^{j-1}Q_i} g_{\mathrm{H},1}\,\mathfrak{m}_i\|_{L^{q_0}(\mathbb{R}^n)}\\
&\lesssim w(2^{j+1}Q_i)^{\frac{1}{p}} |2^{j+1}Q_i|^{-\frac{1}{q_0}} \|\mathfrak{m}_i\|_{L^{p_0}(\mathbb{R}^n)} \left(\int_0^{\ell(Q)} e^{-c\frac{4^{j+i}\ell(Q)^2}{r^2}} r^{-\frac{2n}{p_0}+\frac{2n}{q_0}} \frac{dr}{r}\right)^{\frac{1}{2}}\\
&\lesssim e^{-c4^{j+i}}.
\end{aligned}
$$

Therefore,

$$
\left\|F^1_{1,l}\mathfrak{m}\right\|^p_{L^p(w)} \lesssim \sum_{i\ge 1} \|\mathbf{1}_{2^{l+3}Q_i}F^1_{1,l}\mathfrak{m}_i\|^p_{L^p(w)} + \sum_{i\ge1}\sum_{j\ge l+3} \|\mathbf{1}_{C_j(Q_i)}F^1_{1,l}\mathfrak{m}_i\|^p_{L^p(w)} \lesssim 2^{ln\widehat{r}}. \tag{4.38}
$$

Similarly, noticing that $g_{\mathrm{H},2}$ (disregarding the factor $(\ell(Q)^2/r^2)^{2M}$ since it is controlled by one) is bounded on $L^q(w)$ (see [11]), we get

$$
\left\|\mathbf{1}_{2^{l+3}Q_i}F^2_{1,l}\widetilde{\mathfrak{m}}_i\right\|_{L^p(w)} \lesssim w(2^lQ_i)^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{M}_{p_0}(g_{\mathrm{H},2}\widetilde{\mathfrak{m}}_i)\|_{L^q(w)} \lesssim 2^{l\widehat{r}}2^{-i\varepsilon},
$$

and, since $\{(r^2L)^{M+1}e^{-r^2L}\}_{r>0} \in \mathcal{F}(L^{p_0}-L^{q_0})$, proceeding as before

$$
\begin{aligned}
\left\|\mathbf{1}_{C_j(Q_i)}F^2_{1,l}\widetilde{\mathfrak{m}}_i\right\|_{L^p(w)} &\lesssim w(2^{j+1}Q_i)^{\frac{1}{p}} |2^{j+1}Q_i|^{-\frac{1}{q_0}} \|\widetilde{\mathfrak{m}}_i\|_{L^{p_0}(\mathbb{R}^n)} \left(\int_{\ell(Q)}^{\infty} \left(\frac{\ell(Q)^2}{r^2}\right)^{2M} e^{-c\frac{4^{j+i}\ell(Q)^2}{r^2}} r^{-\frac{2n}{p_0}+\frac{2n}{q_0}} \frac{dr}{r}\right)^{\frac{1}{2}}\\
&\lesssim 2^{-j\left(2M+\frac{n}{p_0}-\frac{n\widehat{r}}{p}\right)}2^{-i(2M+\varepsilon)}.
\end{aligned}
$$

Hence, splitting $\left\|F^2_{1,l}\widetilde{\mathfrak{m}}\right\|_{L^p(w)}$ as in (4.38), and by (4.37), we obtain that $\|F_{1,l}\mathfrak{m}\|_{L^p(w)} \le 2^{ln\widehat{r}}$.

Let us turn to the estimate of $F_{2,l}\mathfrak{m}$. Consider the vertical square function

$$
\mathrm{g}_{\mathrm{H},t}\widetilde{\mathfrak{m}}(x) := \left(\int_{\frac{t}{\sqrt{2}}}^{\infty} \left(\frac{\ell(Q)}{r}\right)^{4M} |(r^2L)^{M+1}e^{-r^2L}\widetilde{\mathfrak{m}}(x)|^2 \frac{dr}{r}\right)^{\frac{1}{2}},
$$

and note that

$$
\begin{aligned}
\|F_{2,l}\mathfrak{m}\|^p_{L^p(w)} \le \sum_{i\ge1}\sum_{j\ge1} &\left(\left\| \mathbf{1}_{C_j(Q_i)} \sup_{(y,t)\in\Gamma(\cdot),\ell(Q)<t\le\frac{2^{j-3}}{2^{l+1}}\ell(Q_i)} \left(\fint_{B(y,2^{l+1}t)} |\mathrm{g}_{\mathrm{H},t}\widetilde{\mathfrak{m}}_i(z)|^{p_0}dz\right)^{\frac{1}{p_0}} \right\|^p_{L^p(w)} \right.\\
&\left. + \left\| \mathbf{1}_{C_j(Q_i)} \sup_{(y,t)\in\Gamma(\cdot),t>\frac{2^{j-3}}{2^{l+1}}\ell(Q_i)} \left(\fint_{B(y,2^{l+1}t)} |\mathrm{g}_{\mathrm{H},t}\widetilde{\mathfrak{m}}_i(z)|^{p_0}dz\right)^{\frac{1}{p_0}} \right\|^p_{L^p(w)} \right)\\
=: &\sum_{i\ge1}\sum_{j\ge1}\left(\|\mathbf{1}_{C_j(Q_i)}F^1_{2,l}\widetilde{\mathfrak{m}}_i\|^p_{L^p(w)} + \|\mathbf{1}_{C_j(Q_i)}F^2_{2,l}\widetilde{\mathfrak{m}}_i\|^p_{L^p(w)}\right)\\
\lesssim &\sum_{i\ge1}\|\mathbf{1}_{2^{l+3}Q_i}F^1_{2,l}\widetilde{\mathfrak{m}}_i\|^p_{L^p(w)} + \sum_{i\ge1}\sum_{j\ge l+3}\|\mathbf{1}_{C_j(Q_i)}F^1_{2,l}\widetilde{\mathfrak{m}}_i\|^p_{L^p(w)}\\
&+ \sum_{i\ge1}\|\mathbf{1}_{2^{l+3}Q_i}F^2_{2,l}\widetilde{\mathfrak{m}}_i\|^p_{L^p(w)} + \sum_{i\ge1}\sum_{j\ge l+3}\|\mathbf{1}_{C_j(Q_i)}F^2_{2,l}\widetilde{\mathfrak{m}}_i\|^p_{L^p(w)}.
\end{aligned} \tag{4.39}
$$

Next, for every $\ell(Q) < t < \sqrt{2}r$ we have that $\mathrm{g}_{\mathrm{H},t}$ is controlled by g_{H} (see (v) in Notation for its definition) which is bounded on $L^q(w)$ (see [11]), hence, for $a = 1, 2$,

$$
\left\|\mathbf{1}_{2^{l+3}Q_i}F^a_{2,l}\widetilde{\mathfrak{m}}_i\right\|_{L^p(w)} \lesssim w(2^lQ_i)^{\frac{1}{p}-\frac{1}{q}}\|\mathcal{M}_{p_0}(g_{\mathrm{H}}\widetilde{\mathfrak{m}}_i)\|_{L^q(w)} \lesssim w(2^lQ_i)^{\frac{1}{p}-\frac{1}{q}}\|\widetilde{\mathfrak{m}}_i\|_{L^q(w)} \lesssim 2^{ln\widehat{r}}2^{-i\varepsilon}.
$$

We observe now that for every $i \geq 1$, $j \geq l+3$, $x \in C_j(Q_i)$, $\ell(Q) < t \leq \frac{2^{j-3}}{2^{l+1}}\ell(Q_i)$, and $(y,t) \in \Gamma(x)$, we have that $B(y, 2^{l+1}t) \subset 2^{j+2}Q_i \setminus 2^{j-1}Q_i$. Therefore, arguing as in the estimates of $\left\|\mathbf{1}_{C_j(Q_i)} F^1_{1,l}\mathfrak{m}_i\right\|_{L^p(w)}$ and $\left\|\mathbf{1}_{C_j(Q_i)} F^2_{1,l}\widetilde{\mathfrak{m}}_i\right\|_{L^p(w)}$, we have that

$$\left\|\mathbf{1}_{C_j(Q_i)} F^1_{2,l}\widetilde{\mathfrak{m}}_i\right\|_{L^p(w)} \lesssim w(2^{j+1}Q_i)^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{M}_{p_0}(\mathbf{1}_{2^{j+1}Q_i\setminus 2^{j-1}Q_i} \mathrm{g}_{\mathrm{H},\ell(Q)}\widetilde{\mathfrak{m}}_i)\|_{L^q(w)} \lesssim 2^{-i(2M+\varepsilon)} 2^{-j\left(2M+\frac{n}{p_0}-\frac{n\widehat{r}}{p}\right)},$$

and

$$\begin{aligned}\left\|\mathbf{1}_{C_j(Q_i)} F^2_{2,l}\widetilde{\mathfrak{m}}_i\right\|_{L^p(w)} &\lesssim w(2^{j+1}Q_i)^{\frac{1}{p}} |2^{j+1}Q_i|^{-\frac{1}{q_0}} \|\widetilde{\mathfrak{m}}_i\|_{L^{p_0}(\mathbb{R}^n)} \left(\int_{\frac{2^{j-3}\ell(Q_i)}{2^{l+1}\sqrt{2}}}^{\infty} \left(\frac{\ell(Q)}{r}\right)^{4M} r^{-2n\left(\frac{1}{p_0}-\frac{1}{q_0}\right)} \frac{dr}{r}\right)^{\frac{1}{2}} \\ &\lesssim 2^{cl} 2^{-i(2M+\varepsilon)} 2^{-j\left(2M+\frac{n}{p_0}-\frac{n\widehat{r}}{p}\right)}.\end{aligned}$$

Consequently, by (4.39) we conclude that $\|F_{2,l}\mathfrak{m}\|_{L^p(w)} \leq 2^{lc}$. Then, in view of (4.36), this and the estimate obtained for $\|F_{1,l}\mathfrak{m}\|_{L^p(w)}$ imply that $\|I\|_{L^p(w)} \leq C$, which finishes the proof. □

4.3.1 Characterization of the weighted Hardy spaces defined by square functions associated with the heat semigroup

Theorem 4.18 follows at once from the following proposition:

Proposition 4.40. *Let $w \in A_\infty$, $q_1, q_2 \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \leq 1$, $\varepsilon > 0$, and $M \in \mathbb{N}$ be such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$. Then,*

(*a*) $\mathbb{H}^p_{L,q_1,\varepsilon,M}(w) = \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)$ *with equivalent norms, for all $m \in \mathbb{N}$.*

(*b*) $H^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)$ *and* $H^p_{\mathcal{S}_{m,\mathrm{H}},q_2}(w)$ *are isomorphic, for all $m \in \mathbb{N}$.*

(*c*) $\mathbb{H}^p_{L,q_1,\varepsilon,M}(w) = \mathbb{H}^p_{\mathrm{G}_{m,\mathrm{H}},q_1}(w) = \mathbb{H}^p_{\mathcal{G}_{m,\mathrm{H}},q_1}(w)$, *with equivalent norms, for all $m \in \mathbb{N}_0$.*

Proof of Proposition 4.40, part (*a*).

Fix $w \in A_\infty$, $q_1 \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \leq 1$ $\varepsilon > 0$, and $m, M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$.

For all $f \in \mathbb{H}^p_{L,q_1,\varepsilon,M}(w)$, applying Proposition 4.22, we obtain that

$$\|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q_1,\varepsilon,M}(w)}. \tag{4.41}$$

Then, since in particular $f \in L^{q_1}(w)$, we conclude that $f \in \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)$, and hence $\mathbb{H}^p_{L,q_1,\varepsilon,M}(w) \subset \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)$.

As for proving the converse, we shall show for all $f \in \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)$ that we can find a $(w, q_1, p, \varepsilon, M)$-representation of f, i.e. $f = \sum_{i=1}^\infty \lambda_i \mathfrak{m}_i$, such that

$$\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{\frac{1}{p}} \lesssim \|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)}.$$

Following some ideas of [55, Lemma 4.2], for each $l \in \mathbb{Z}$ and for some $0 < \gamma < 1$ to be chosen later, set

$$O_l := \{x \in \mathbb{R}^n : \mathcal{S}_{m,\mathrm{H}} f(x) > 2^l\}, \quad E_l := \mathbb{R}^n \setminus O_l, \quad E_l^* := \left\{x \in \mathbb{R}^n : \frac{|E_l \cap B(x,r)|}{|B(x,r)|} \geq \gamma, \text{ for all } r > 0\right\},$$

and $O_l^* := \mathbb{R}^n \setminus E_l^* = \left\{x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_l})(x) > 1 - \gamma\right\}$. Recall that $\mathcal{M}$ is the centered Hardy-Littlewood maximal operator. We have that O_l and O_l^* are open, and that $O_{l+1} \subseteq O_l$, $O_{l+1}^* \subseteq O_l^*$, and $O_l \subseteq O_l^*$. Besides, since

$w \in A_\infty$ then $\mathcal{M} : L^r(w) \to L^{r,\infty}(w)$, for every $r > r_w$. Also $\|\mathcal{S}_{m,\mathrm{H}} f\|_{L^{q_1}(w)} \lesssim \|f\|_{L^{q_1}(w)} < \infty$, because $q_1 \in \mathcal{W}_w(p_-(L), p_+(L))$ (see Theorem 3.1). Hence,

$$w(O_l^*) \le c_\gamma w(O_l) \le \frac{1}{2^{lq_1}} \|\mathcal{S}_{m,\mathrm{H}} f\|^{q_1}_{L^{q_1}(w)} \lesssim \frac{1}{2^{lq_1}} \|f\|^{q_1}_{L^{q_1}(w)}, \quad \forall\, l \in \mathbb{Z} \tag{4.42}$$

which implies that E_l^* cannot be empty. Therefore, for each l, we can take a Whitney decomposition, $\{Q_l^j\}_{j\in\mathbb{N}}$, of O_l^*:

$$O_l^* = \bigcup_{j\in\mathbb{N}} Q_l^j, \qquad \operatorname{diam}(Q_l^j) \le d(Q_l^j, \mathbb{R}^n \setminus O_l^*) \le 4\operatorname{diam}(Q_l^j),$$

and the cubes Q_l^j have disjoint interiors. Finally, define, for each $j \in \mathbb{N}$ and $l \in \mathbb{Z}$, the sets

$$T_l^j := (Q_l^j \times (0,\infty)) \bigcap \left(\widehat{O_l^*} \setminus \widehat{O_{l+1}^*}\right), \tag{4.43}$$

where $\widehat{O_l^*} := \left\{(x,t) \in \mathbb{R}^{n+1}_+ : d(x, \mathbb{R}^n \setminus O_l^*) \ge t\right\}$ and we denote $\mathcal{R}(E_l^*) := \mathbb{R}^n \setminus \widehat{O_l^*}$. Proceeding as in the proof of Theorem 2.76 when, following the notation there, we showed that $\operatorname{supp} f \subset \left(\bigcup_{k\in\mathbb{Z}} \widehat{O_k^*} \setminus \widehat{O_{k+1}^*}\right) \bigcup \mathbb{F}$ with $\mathbb{F} \subset \mathbb{R}^{n+1}_+$ and $\int_0^\infty \int_{\mathbb{R}^n} \mathbf{1}_{\mathbb{F}}(y,t) \frac{dy\,dt}{t^{n+1}} = 0$. We can show that

$$\operatorname{supp} T_s f(x) := \operatorname{supp} (s^2 L)^m e^{-s^2 L} f(x) \subset \left(\bigcup_{l\in\mathbb{Z}} \left(\widehat{O_l^*} \setminus \widehat{O_{l+1}^*}\right)\right) \bigcup \mathbb{F} = \left(\bigcup_{l\in\mathbb{Z},\, j\in\mathbb{N}} T_l^j\right) \bigcup \mathbb{F} \tag{4.44}$$

with $\mathbb{F} \subset \mathbb{R}^{n+1}_+$ and $\mu(\mathbb{F}) := \iint_{\mathbb{R}^{n+1}_+} \mathbf{1}_{\mathbb{F}}(y,s) \frac{dy\,ds}{s} = 0$. Then, if we consider the following Calderón reproducing formula of $f \in L^{q_1}(w)$:

$$f(x) = \widetilde{C} \int_0^\infty \left((t^2L)^m e^{-t^2L}\right)^{M+2} f(x) \frac{dt}{t} = \widetilde{C} \lim_{N\to\infty} \int_{N^{-1}}^N \left((t^2L)^m e^{-t^2L}\right)^{M+2} f(x) \frac{dt}{t},$$

with the integral converging in $L^{q_1}(w)$ (in order to justify this equalities we note that the vertical square function defined by $\left((t^2L)^m e^{-t^2L}\right)^{M+1}$, $M \in \mathbb{N}_0$, is bounded on $L^{\widetilde{p}}(w)$ for all $\widetilde{p} \in \mathcal{W}_w(p_-(L), p_+(L))$, and follow a similar explanation to that of Remark 3.49), by (4.44), we obtain that

$$\begin{aligned} f(x) &= \widetilde{C} \int_0^\infty \left((t^2L)^m e^{-t^2L}\right)^{M+1} \Big(\sum_{j\in\mathbb{N}, l\in\mathbb{Z}} \mathbf{1}_{T_l^j}(\cdot,t) (t^2L)^m e^{-t^2L} f(\cdot)\Big)(x) \frac{dt}{t} \\ &= \widetilde{C} \lim_{N\to\infty} \int_{N^{-1}}^N \left((t^2L)^m e^{-t^2L}\right)^{M+1} \Big(\sum_{j\in\mathbb{N}, l\in\mathbb{Z}} \mathbf{1}_{T_l^j}(\cdot,t) (t^2L)^m e^{-t^2L} f(\cdot)\Big)(x) \frac{dt}{t}, \end{aligned} \tag{4.45}$$

in $L^{q_1}(w)$. Now, we set

$$\lambda_l^j := 2^l w(Q_l^j)^{\frac{1}{p}} \quad \text{and} \quad \mathfrak{m}_l^j(x) := \frac{1}{\lambda_l^j} \int_0^\infty \left((t^2L)^m e^{-t^2L}\right)^{M+1} \left(f_{l,m}^j(\cdot,t)\right)(x) \frac{dt}{t},$$

where $f_{l,m}^j(x,t) := \mathbf{1}_{T_l^j}(x,t)(t^2L)^m e^{-t^2L} f(x)$. We will show that

$$\sum_{j\in\mathbb{N}, l\in\mathbb{Z}} \widetilde{C} \lambda_l^j \mathfrak{m}_l^j \quad \text{is a} \quad (w, q_1, p, \varepsilon, M)\text{-representation of } f. \tag{4.46}$$

We start showing that there exists a uniform constant C_0, such that $C_0^{-1} \mathfrak{m}_l^j$ is a $(w, q_1, p, \varepsilon, M)$-molecule, for all $j \in \mathbb{N}$ and $l \in \mathbb{Z}$. To this end, we estimate, for all $0 \le k \le M$, $1 \le i$, $j \in \mathbb{N}$, and $l \in \mathbb{Z}$, the $L^{q_1}(w)$ norms of the functions $(\ell(Q_l^j)^2 L)^{-k} \mathfrak{m}_l^j \mathbf{1}_{C_i(Q_l^j)}$. Before that, we set

$$\mathfrak{R}^{\ell(Q_l^j)}(E_{l+1}^*) := \{(y,t) \in \mathcal{R}(E_{l+1}^*) : y \in Q_l^j,\ 0 < t \le 5\sqrt{n}\,\ell(Q_l^j)\},$$

and note that, since for all $(y,t) \in T_l^j$, we have that

$$t \le d(y, \mathbb{R}^n \setminus O_l^*) \le d(Q_l^j, \mathbb{R}^n \setminus O_l^*) + \operatorname{diam}(Q_l^j) \le 5\operatorname{diam}(Q_l^j),$$

we conclude that,

$$T_l^j \subset \Re^{\ell(Q_l^j)}(E_{l+1}^*). \tag{4.47}$$

Then, for all $(y,t) \in T_l^j$ and $c = 11\sqrt{n}$,

$$B(y,t) \subset cQ_l^j. \tag{4.48}$$

Now, by definition of T_l^j, we have that for every $(y,t) \in T_l^j$ there exists $y_0 \in E_{l+1}^*$ such that $|E_{l+1} \cap B(y_0,t)| \ge \gamma|B(y_0,t)|$ and $|y_0 - y| < t$. Besides, considering $z := y - \frac{t(y-y_0)}{2|y-y_0|}$, we have that $B\left(z, \frac{t}{2}\right) \subset B(y_0,t) \cap B(y,t)$. Consequently,

$$\begin{aligned}\gamma|B(y_0,t)| &\le |E_{l+1} \cap B(y_0,t)| \le |E_{l+1} \cap B(y,t)| + |B(y_0,t) \setminus B(y,t)| \\ &\le |E_{l+1} \cap B(y,t)| + \left|B(y_0,t) \setminus B\left(z, \frac{t}{2}\right)\right| = |E_{l+1} \cap B(y,t)| + |B(y_0,t)|\left(1 - \frac{1}{2^n}\right).\end{aligned}$$

Then, for $\gamma = 1 - \frac{1}{2^{n+1}}$, we obtain

$$t^n \lesssim |E_{l+1} \cap B(y,t)|. \tag{4.49}$$

We are now ready to consider the case $i = 1$. For every $t > 0$, let $\mathcal{T}_t := (t^2L)^{mM+m-k}e^{-t^2(M+1)L}$, and for every $h \in L^{q_1'}(w^{1-q_1'})$ write $Q_L h(x,t) := \mathcal{T}_t^* h(x)$, with $(x,t) \in \mathbb{R}^{n+1}_+$. As in Remark 3.49 one can show that $Q_L : L^{q_1'}(w^{1-q_1'}) \to L^{q_1'}_{\mathbb{H}}(w^{1-q_1'})$, since $q_1' \in \mathcal{W}_{w^{1-q_1'}}(p_-(L^*), p_+(L^*))$. Hence its adjoint Q_L^* has a bounded extension from $L^{q_1}_{\mathbb{H}}(w)$ to $L^{q_1}(w)$, where

$$Q_L^* h(x) = \int_0^\infty \mathcal{T}_t h(x,t)\frac{dt}{t} = \int_0^\infty (t^2L)^{mM+m-k}e^{-t^2(M+1)L}h(x,t)\frac{dt}{t}.$$

Here, as in Remark 3.49, $\mathcal{T}_t h(x,t) = \mathcal{T}_t\big(h(\cdot,t)\big)(x)$, for $(x,t) \in \mathbb{R}^{n+1}_+$, and abusing notation L denotes the infinitesimal generator of e^{-tL} on $L^{q_1}(w)$, see [11, Remark 3.5] (recall that $q_1 \in \mathcal{W}_w(p_-(L), p_+(L))$). Now, consider $\widetilde{g}(x,t) := t^{2k} f_{l,m}^j(x,t) \in L^{q_1}_{\mathbb{H}}(w)$ (this follows by (4.47) and by the boundedness on $L^{q_1}(w)$ of the vertical square function defined by $(t^2L)^m e^{-t^2L}$, see [11]), and

$$\mathcal{I} := \left\{h \in L^{q_1'}(w^{1-q_1'}) : \|h\|_{L^{q_1'}(w^{1-q_1'})} = 1,\ \operatorname{supp} h \subset 4Q_l^j\right\}.$$

Then, from (4.48), (4.49), and (4.47) we obtain

$$\begin{aligned}\left\|((\ell(Q_l^j)^2L)^{-k}\mathbf{m}_l^j)\mathbf{1}_{4Q_l^j}\right\|_{L^{q_1}(w)} &= \frac{\ell(Q_l^j)^{-2k}}{\lambda_l^j}\left\|\int_0^\infty (t^2L)^{mM+m-k}e^{-t^2(M+1)L}\widetilde{g}(\cdot,t)\frac{dt}{t}\right\|_{L^{q_1}(4Q_l^j,w)} \\ &= \frac{\ell(Q_l^j)^{-2k}}{\lambda_l^j}\sup_{h\in\mathcal{I}}\left|\int_{\mathbb{R}^n} Q_L^*\widetilde{g}(y)\cdot h(y)\,dy\right| \\ &= \frac{\ell(Q_l^j)^{-2k}}{\lambda_l^j}\sup_{h\in\mathcal{I}}\left|\int_{\mathbb{R}^n}\int_0^\infty \widetilde{g}(y,t)\cdot\mathcal{T}_t^*h(y)\,\frac{dt\,dy}{t}\right| \\ &\lesssim \frac{\ell(Q_l^j)^{-2k}}{\lambda_l^j}\sup_{h\in\mathcal{I}}\iint_{T_l^j} t^{2k}\left|(t^2L)^m e^{-t^2L}f(y)\cdot\mathcal{T}_t^*h(y)\right|\int_{B(y,t)\cap E_{l+1}} dx\frac{dt}{t^{n+1}}dy\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{\lambda_l^j} \sup_{h\in\mathcal{I}} \int_{cQ_l^j\cap E_{l+1}} \iint_{\Gamma(x)} \left|(t^2L)^m e^{-t^2L} f(y)\cdot \mathcal{T}_t^* h(y)\right| \frac{dy\,dt}{t^{n+1}} dx \\
&\leq \frac{1}{\lambda_l^j} \left\|\mathcal{S}_{m,\mathrm{H}} f\right\|_{L^{q_1}(cQ_l^j\cap E_{l+1},w)} \sup_{h\in\mathcal{I}} \left\| \left|\!\left|\!\left| \mathcal{T}_t^* h \right|\!\right|\!\right|_{\Gamma(\cdot)} \right\|_{L^{q_1'}(w^{1-q_1'})} \\
&\leq \frac{1}{\lambda_l^j} w(Q_l^j)^{\frac{1}{p}} 2^l \sup_{h\in\mathcal{I}} \left\| \left|\!\left|\!\left| \mathcal{T}_t^* h \right|\!\right|\!\right|_{\Gamma(\cdot)} \right\|_{L^{q_1'}(w^{1-q_1'})} \\
&= w(Q_l^j)^{\frac{1}{p}-1} \sup_{h\in\mathcal{I}} \left\| \left|\!\left|\!\left| \mathcal{T}_t^* h \right|\!\right|\!\right|_{\Gamma(\cdot)} \right\|_{L^{q_1'}(w^{1-q_1'})},
\end{aligned}
\tag{4.50}
$$

where in the last inequality we have used that $\mathcal{S}_{m,\mathrm{H}} f(x) \leq 2^{l+1}$ for every $x \in E_{l+1}$. In order to estimate the term with the supremum, we fix $h \in \mathcal{I}$ and note that changing variable t into $\frac{t}{\sqrt{M+1}}$ and using Proposition 2.60,

$$
\begin{aligned}
\left\| \left|\!\left|\!\left| \mathcal{T}_t^* h \right|\!\right|\!\right|_{\Gamma(\cdot)} \right\|_{L^{q_1'}(w^{1-q_1'})} &= C_M \left\| \left|\!\left|\!\left| \mathcal{T}^*_{\frac{t}{\sqrt{M+1}}} h \right|\!\right|\!\right|_{\Gamma^{\frac{1}{\sqrt{M+1}}}(\cdot)} \right\|_{L^{q_1'}(w^{1-q_1'})} \lesssim C_M \left\| \left|\!\left|\!\left| \mathcal{T}^*_{\frac{t}{\sqrt{M+1}}} h \right|\!\right|\!\right|_{\Gamma(\cdot)} \right\|_{L^{q_1'}(w^{1-q_1'})} \\
&= \left\| \left(\iint_{\Gamma(\cdot)} \left|(t^2L^*)^{mM+m-k} e^{-t^2L^*} h(y)\right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}} \right\|_{L^{q_1'}(w^{1-q_1'})} \lesssim \|h\|_{L^{q_1'}(w^{1-q_1'})} = 1,
\end{aligned}
$$

where last estimate holds since $q_1' \in \mathcal{W}_{w^{1-q_1'}}(p_-(L^*), p_+(L^*))$ (see Theorem 3.1). Plugging this into (4.50) we conclude that

$$
\left\| \left((\ell(Q_l^j)^2 L)^{-k} \mathfrak{m}_l^j \right) \mathbf{1}_{4Q_l^j} \right\|_{L^{q_1}(w)} \lesssim w(Q_l^j)^{\frac{1}{q_1}-1}. \tag{4.51}
$$

We consider now $i \geq 2$. Note that since $w \in RH_{\left(\frac{p_+(L)}{q_1}\right)'}$ there exists q_0, $\max\{2, q_1\} < q_0 < p_+(L)$, such that $w \in RH_{\left(\frac{q_0}{q_1}\right)'}$. Then,

$$
\begin{aligned}
&\left\| \left((\ell(Q_l^j)^2 L)^{-k} \mathfrak{m}_l^j \right) \mathbf{1}_{C_i(Q_l^j)} \right\|_{L^{q_1}(w)} \leq \frac{1}{\lambda_l^j} \left\| \int_0^\infty \left| (\ell(Q_l^j)^2 L)^{-k} \left((t^2L)^m e^{-t^2L} \right)^{M+1} \left(f_{l,m}^j(\cdot,t) \right) \right| \frac{dt}{t} \right\|_{L^{q_1}(C_i(Q_l^j),w)} \\
&\lesssim \frac{\ell(Q_l^j)^{-2k}}{\lambda_l^j} w(2^{i+1}Q_l^j)^{\frac{1}{q_1}} |2^{i+1}Q_l^j|^{-\frac{1}{q_0}} \left\| \int_0^\infty \left| t^{2k} (t^2L)^{mM+m-k} e^{-t^2(M+1)L} \left(f_{l,m}^j(\cdot,t) \right) \right| \frac{dt}{t} \right\|_{L^{q_0}(C_i(Q_l^j))}.
\end{aligned}
$$

Apply Minkowski's inequality, the fact that $\{(t^2L)^m e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^{q_0})$, and (4.47) we have the following estimate for the last integral above:

$$
\begin{aligned}
&\left\| \int_0^\infty \left| t^{2k} (t^2L)^{mM+m-k} e^{-t^2(M+1)L} \left(f_{l,m}^j(\cdot,t) \right) \right| \frac{dt}{t} \right\|_{L^{q_0}(C_i(Q_l^j))} \\
&\quad \leq \int_0^\infty t^{2k} \left\| (t^2L)^{mM+m-k} e^{-t^2(M+1)L} \left(\mathbf{1}_{Q_l^j}(\cdot) f_{l,m}^j(\cdot,t) \right) \right\|_{L^{q_0}(C_i(Q_l^j))} \frac{dt}{t} \\
&\quad \lesssim \int_0^{5\sqrt{n}\ell(Q_l^j)} \left(\int_{Q_l^j} \left| f_{l,m}^j(y,t) \right|^2 dy \right)^{\frac{1}{2}} t^{2k} t^{-n\left(\frac{1}{2}-\frac{1}{q_0}\right)} e^{-c\frac{4^i \ell(Q_l^j)^2}{t^2}} \frac{dt}{t} \\
&\quad \lesssim \ell(Q_l^j)^{2k} \left(\iint_{T_l^j} \left| (t^2L)^m e^{-t^2L} f(y) \right|^2 \frac{dy\,dt}{t} \right)^{\frac{1}{2}} \left(\int_0^{5\sqrt{n}\ell(Q_l^j)} t^{-2n\left(\frac{1}{2}-\frac{1}{q_0}\right)} e^{-c\frac{4^i \ell(Q_l^j)^2}{t^2}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\quad =: II_1 \times II_2.
\end{aligned}
$$

For II_1, we proceed as in the estimate of I_1 and obtain

$$II_1 \lesssim \ell(Q_l^j)^{2k}\left(\iint_{T_l^j}\left|(t^2L)^m e^{-t^2L}f(y)\right|^2\int_{B(y,t)\cap E_{l+1}} dx\,dy\frac{dt}{t^{n+1}}\right)^{\frac12}$$
$$\lesssim \ell(Q_l^j)^{2k}\left\|\left|\left|\left|(t^2L)^m e^{-t^2L}f\right|\right|\right|_{\Gamma(\cdot)}\right\|_{L^2(cQ_l^j\cap E_{l+1})} \lesssim \ell(Q_l^j)^{2k}|Q_l^j|^{\frac12}2^l.$$

As for the estimate of II_2, changing the variable t into $\frac{2^i\ell(Q_l^j)}{t}$ we get

$$II_2 \lesssim (2^i\ell(Q_l^j))^{-n\left(\frac12-\frac{1}{q_0}\right)}e^{-c4^i}\left(\int_0^\infty t^{2n\left(\frac12-\frac{1}{q_0}\right)}e^{-ct^2}\frac{dt}{t}\right)^{\frac12} \lesssim (2^i\ell(Q_l^j))^{-n\left(\frac12-\frac{1}{q_0}\right)}e^{-c4^i}.$$

Hence, for $i\ge 2$, by (1.11),

$$\left\|\left((\ell(Q_l^j)^2L)^{-k}\mathfrak{m}_l^j\right)\mathbf{1}_{C_i(Q_l^j)}\right\|_{L^{q_1}(w)} \lesssim \frac{1}{\lambda_l^j}e^{-c4^i}2^{-\frac{in}{2}}2^l w(2^{i+1}Q_l^j)^{\frac{1}{q_1}} \lesssim e^{-c4^i}w(2^{i+1}Q_l^j)^{\frac{1}{q_1}-\frac1p}.$$

From this and (4.51), we infer that there exists a constant $C_0>0$ such that, for all $j\in\mathbb{N}$ and $l\in\mathbb{Z}$, $\|\mathfrak{m}_l^j\|_{mol,w}\le C_0$. Therefore, for every $j\in\mathbb{N}$ and $l\in\mathbb{Z}$, we have that $C_0^{-1}\mathfrak{m}_l^j$ are (w,q_1,p,ε,M)-molecules associated with the cubes Q_l^j.

Let us now prove that $\{\lambda_l^j\}_{j\in\mathbb{N},l\in\mathbb{Z}}\in\ell^p$. Since for each $l\in\mathbb{Z}$, $\{Q_j^l\}_{j\in\mathbb{N}}$ is a Whitney decomposition of O_l^*, by (4.42), and since $f\in\mathbb{H}^p_{S_{m,\mathrm{H}},q_1}(w)$, we obtain

$$\begin{aligned}\left(\sum_{j\in\mathbb{N},l\in\mathbb{Z}}|\lambda_l^j|^p\right)^{\frac1p} &= \left(\sum_{j\in\mathbb{N},l\in\mathbb{Z}}2^{pl}w(Q_l^j)\right)^{\frac1p} = \left(\sum_{l\in\mathbb{Z}}2^{pl}w(O_l^*)\right)^{\frac1p} \lesssim \left(\sum_{l\in\mathbb{Z}}2^{pl}w(O_l)\right)^{\frac1p}\\ &\lesssim \left(\sum_{l\in\mathbb{Z}}\int_{2^{l-1}}^{2^l}\lambda^p w(O_l)\frac{d\lambda}{\lambda}\right)^{\frac1p} \lesssim \left(\int_0^\infty \lambda^p w(\{x\in\mathbb{R}^n: S_{m,\mathrm{H}}f(x)>\lambda\})\frac{d\lambda}{\lambda}\right)^{\frac1p} = C\|S_{m,\mathrm{H}}f\|_{L^p(w)}<\infty. \end{aligned}\tag{4.52}$$

Thus to conclude (4.46), we finally show that

$$f=\sum_{j\in\mathbb{N},l\in\mathbb{Z}}\widetilde{C}\lambda_l^j\mathfrak{m}_l^j \quad\text{in}\quad L^{q_1}(w).\tag{4.53}$$

To that end, we consider the operator $\mathcal{T}_{t,L}^M := ((t^2L)^m e^{-t^2L})^{M+1}$, $M\ge 0$, whose adjoint (in $L^2(\mathbb{R}^n)$) has the expression $(\mathcal{T}_{t,L}^M)^* = ((t^2L^*)^m e^{-t^2L^*})^{M+1} =: \mathcal{T}_{t,L^*}^M$, and $Q_L^M f(x,t) := \mathcal{T}_{t,L^*}^M f(x)$ for $(x,t)\in\mathbb{R}^{n+1}_+$ and $f\in L^2(\mathbb{R}^n)$. Then, proceeding as in Remark 3.49, using that the vertical square function defined by $\mathcal{T}_{t,L^*}^M$ is bounded on $L^{q_1'}(w^{1-q_1'})$, we can obtain that $Q_L^M: L^{q_1'}(w^{1-q_1'})\to L^{q_1'}_{\mathbb{H}}(w^{1-q_1'})$, for all $q_1'\in\mathcal{W}_{w^{1-q_1'}}(p_-(L^*),p_+(L^*))$; and then, its adjoint $(Q_L^M)^*: L^{q_1}_{\mathbb{H}}(w)\to L^{q_1}(w)$, where

$$(Q_L^M)^*h(x)=\int_0^\infty ((t^2L)^m e^{-t^2L})^{M+1}h(x,t)\frac{dt}{t},$$

understanding again that by abuse of notation L denotes the infinitesimal generator of e^{-tL} on $L^{q_1}(w)$ (see [11, Remark 3.5]).

Consequently, recalling that $f_{l,m}^j(x,t) = \mathbf{1}_{T_l^j}(x,t)(t^2L)^m e^{-t^2L}f(x)$ where the sets $\{T_l^j\}_{j\in\mathbb{N},l\in\mathbb{Z}}$ are pairwise disjoint, it follows that

$$\left\|\sum_{j\in\mathbb{N},l\in\mathbb{Z}}f_{l,m}^j\right\|_{L^{q_1}_{\mathbb{H}}(w)} = \left\|\sum_{j\in\mathbb{N},l\in\mathbb{Z}}|f_{l,m}^j|\right\|_{L^{q_1}_{\mathbb{H}}(w)} \le \left\|\left(\int_0^\infty |(t^2L)^m e^{-t^2L}f|^2\frac{dt}{t}\right)^{\frac12}\right\|_{L^{q_1}(w)} \lesssim \|f\|_{L^{q_1}(w)}.$$

Hence, by (4.45) the observations made above, and the dominated convergence theorem we obtain

$$\begin{aligned}\Bigg\| f - \sum_{j+|l|\le K} \widetilde{C}\lambda_l^j \mathfrak{m}_l^j \Bigg\|_{L^{q_1}(w)} &= \widetilde{C}\,\Bigg\| (Q_L^M)^* \Bigg(\sum_{j+|l|>0} f_{l,m}^j\Bigg) - \sum_{j+|l|\le K} (Q_L^M)^* f_{l,m}^j \Bigg\|_{L^{q_1}(w)} \\ &= \widetilde{C}\,\Bigg\| (Q_L^M)^* \Bigg(\sum_{j+|l|>K} f_{l,m}^j\Bigg)\Bigg\|_{L^{q_1}(w)} \lesssim \Bigg\| \sum_{j+|l|>K} f_{l,m}^j \Bigg\|_{L^{q_1}_{\mathbb{H}}(w)} \longrightarrow 0, \qquad \text{as } K \to \infty. \quad (4.54)\end{aligned}$$

This proves (4.53) and therefore, $\sum_{j+|l|>0} \lambda_l^j \mathfrak{m}_l^j$ is a $(w, q_1, p, \varepsilon, M)$ representation of f such that

$$\Bigg(\sum_{j+|l|>0} |\lambda_l^j|^p\Bigg)^{\frac{1}{p}} \lesssim \|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)}.$$

Consequently, $f \in \mathbb{H}^p_{L,q_1,\varepsilon,M}(w)$ and $\|f\|_{\mathbb{H}^p_{L,q_1,\varepsilon,M}(w)} \lesssim \|\mathcal{S}_{m,\mathrm{H}} f\|_{L^p(w)}$, which completes the proof. □

Proof of Proposition 4.40, part (*b*).

Fix $w \in A_\infty$, $q_1, q_2 \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \le 1$ $\varepsilon > 0$, and $m, M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$.

For $f \in \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)$ consider the $(w, q_1, p, \varepsilon, M)$−representation of f, ($f = \sum_{j+|l|>0} \lambda_l^j \mathfrak{m}_l^j$) obtained in the proof of Proposition 4.40, part (*a*). Then define for each $N \in \mathbb{N}$

$$f_N := \sum_{0<j+|l|\le N} \lambda_l^j \mathfrak{m}_l^j.$$

We have that, for each $N \in \mathbb{N}$, $f_N, f - f_N \in \mathbb{H}^p_{L,q_1,\varepsilon,M}(w) = \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)$. Moreover, since $\sum_{j+|l|>N+1} \lambda_l^j \mathfrak{m}_l^j$ is a $(w, q_1, p, \varepsilon, M)$−representation of $f - f_N$, we have

$$\|\mathcal{S}_{m,\mathrm{H}}(f - f_N)\|_{L^p(w)} = \|f - f_N\|_{\mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_1}(w)} \lesssim \|f - f_N\|_{\mathbb{H}^p_{L,q_1,\varepsilon,M}(w)} \le \Bigg(\sum_{j+|l|>N+1} |\lambda_l^j|^p\Bigg)^{\frac{1}{p}} \underset{N\to\infty}{\longrightarrow} 0.$$

Consequently in order to conclude that $f \in H^p_{\mathcal{S}_{m,\mathrm{H}},q_2}(w)$, it is enough to show that, for each $N \in \mathbb{N}$, $f_N \in \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_2}(w)$, or equivalently that $f_N \in \mathbb{H}^p_{L,q_2,\varepsilon,M}(w)$. Let us see the latter. For every N, following the same computations done in the proof of part (*a*) to show that the $\mathfrak{m}_l^j$ are $(w, q, p, \varepsilon, M)$-molecules, but replacing the $L^q(w)$ norm with the $L^{q_2}(w)$ norm, we obtain that, for all $i, j \in \mathbb{N}$, $l \in \mathbb{Z}$, and $0 \le k \le M$,

$$\|(\ell(Q_l^j)^2 L)^{-k} \mathfrak{m}_l^j\|_{L^{q_2}(C_i(Q_l^j),w)} \lesssim e^{-c4^i} w(2^{i+1} Q_l^j)^{\frac{1}{q_2} - \frac{1}{p}}.$$

Hence, $\mathfrak{m}_l^j$ is a multiple of a $(w, q_2, p, \varepsilon, M)$−molecule. Besides, using (1.11),

$$\begin{aligned}\|f_N\|^p_{L^{q_2}(w)} &\lesssim \sum_{i\ge1} \sum_{0<j+|l|\le N} |\lambda_l^j|^p\, \|\mathfrak{m}_l^j\|^p_{L^{q_2}(C_i(Q_l^j),w)} \\ &\lesssim \sum_{i\ge1} \sum_{0<j+|l|\le N} |\lambda_l^j|^p\, e^{-c4^i} w(2^{i+1} Q_l^j)^{\frac{p}{q_2}-1} \lesssim \sum_{0<j+|l|\le N} 2^{pl} w(Q_l^j)^{\frac{p}{q_2}} \\ &\lesssim \delta_N^{\frac{p}{q_2}-1} \sum_{0<j+|l|\le N} 2^{pl} w(Q_l^j) \lesssim \delta_N^{\frac{p}{q_2}-1} \|\mathcal{S}_{m,\mathrm{H}} f\|^p_{L^p(w)} < \infty.\end{aligned}$$

where $\delta_N := \min_{0<j+|l|\le N} w(Q_l^j)$. Then, for each $N \in \mathbb{N}$, $\sum_{0<j+|l|\le N} \lambda_l^j \mathfrak{m}_l^j$ is a $(w, q_2, p, \varepsilon, M)$-representation of f_N. Hence, $\{f_N\}_{N\in\mathbb{N}} \subset \mathbb{H}^p_{L,q_2,\varepsilon,M}(w) = \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{H}},q_2}(w)$. □

Proof of Proposition 4.40, part (c).

For $f \in \mathbb{H}^p_{\mathcal{G}_{m,\mathrm{H}},q_1}(w)$, applying Theorem 3.23, part (a), and the fact that $\mathrm{G}_{m,\mathrm{H}}f(x) \le \mathcal{G}_{m,\mathrm{H}}f(x)$ for every $x \in \mathbb{R}^n$ and for every $m \in \mathbb{N}_0$, we conclude

$$\|\mathcal{S}_{m+1,\mathrm{H}}f\|_{L^p(w)} \lesssim \|\mathrm{G}_{m,\mathrm{H}}f\|_{L^p(w)} \le \|\mathcal{G}_{m,\mathrm{H}}f\|_{L^p(w)}.$$

This, together with Proposition 4.40, part (a), implies

$$\mathbb{H}^p_{\mathcal{G}_{m,\mathrm{H}},q_1}(w) \subset \mathbb{H}^p_{\mathrm{G}_{m,\mathrm{H}},q_1}(w) \subset \mathbb{H}^p_{\mathcal{S}_{m+1,\mathrm{H}},q_1}(w) = \mathbb{H}^p_{L,q_1,\varepsilon,M}(w).$$

To finish the proof, take $f \in \mathbb{H}^p_{L,q_1,\varepsilon,M}(w)$. Then, by Proposition 4.22, we have that

$$\|\mathcal{G}_{m,\mathrm{H}}f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q_1,\varepsilon,M}(w)}.$$

Consequently, $\mathbb{H}^p_{L,q_1,\varepsilon,M}(w) \subset \mathbb{H}^p_{\mathcal{G}_{m,\mathrm{H}},q_1}(w)$. □

4.3.2 Characterization of the weighted Hardy spaces defined by square functions associated with the Poisson semigroup

In this section, we prove Theorem 4.19, which is obtained as a consequence of the following proposition.

Proposition 4.55. *Given $w \in A_\infty$, $q_1, q_2 \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \le 1$, $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{2}\right)$, $K \in \mathbb{N}$, and $\varepsilon_0 = 2M + 2K + \frac{n}{2} - \frac{nr_w}{p}$, there hold,*

(a) $\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w) = \mathbb{H}^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w)$, *with equivalent norms.*

(b) $H^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w)$ *and* $H^p_{\mathcal{S}_{K,\mathrm{P}},q_2}(w)$ *are isomorphic.*

(c) $\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w) = \mathbb{H}^p_{\mathrm{G}_{K-1,\mathrm{P}},q_1}(w) = \mathbb{H}^p_{\mathcal{G}_{K-1,\mathrm{P}},q_1}(w)$, *with equivalent norms.*

Proof of Proposition 4.55, part (a).

To prove the left-to-the-right inclusion observe that if $f \in \mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w)$, in particular $f \in L^{q_1}(w)$, and from Proposition 4.22, part (b), we have that

$$\|\mathcal{S}_{K,\mathrm{P}}f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w)}.$$

Therefore, we conclude that $\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w) \subset \mathbb{H}^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w)$.

As for proving the converse, take $f \in \mathbb{H}^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w)$ and define the same sets, (O_l, O_l^*, T_l^j, etc), defined in the proof of Proposition 4.40, part (a), but replacing $\mathcal{S}_{m,\mathrm{H}}$ with $\mathcal{S}_{K,\mathrm{P}}$. Besides, consider the following Calderón reproducing formula of f (in order to justify it we follow a similar explanation to that of Remark 3.49),

$$f(x) = C\int_0^\infty \left(\left((t^2L)^{M+K}e^{-t\sqrt{L}}\right)^2 f(\cdot)\right)(x)\frac{dt}{t} = C\lim_{N\to\infty}\int_{N^{-1}}^N (t^2L)^{2M+K}e^{-t\sqrt{L}}\left((t^2L)^K e^{-t\sqrt{L}}f(\cdot)\right)(x)\frac{dt}{t}.$$

Again following the same computations as in the proof of Proposition 4.40, part (a), considering $T_sf(x) := (t^2L)^K e^{-t\sqrt{L}}f(x)$, we can show that $\operatorname{supp} T_sf(x) \subset \left(\bigcup_{l\in\mathbb{Z}} \widehat{O_l^*} \setminus \widehat{O_{l+1}^*}\right) \bigcup \mathbb{F}$. Consequently, we have that

$$f(x) = C\lim_{N\to\infty}\int_{N^{-1}}^N (t^2L)^{2M+K}e^{-t\sqrt{L}}\left(\sum_{j\in\mathbb{N}, l\in\mathbb{Z}} \mathbf{1}_{T_l^j}(\cdot,t)(t^2L)^K e^{-t\sqrt{L}}f(\cdot)\right)(x)\frac{dt}{t},$$

in $L^{q_1}(w)$. Hence, considering

$$\lambda_l^j := 2^l w(Q_l^j)^{\frac{1}{p}}, \quad \text{and} \quad \mathfrak{m}_l^j(x) := \frac{1}{\lambda_l^j}\int_0^\infty (t^2L)^{2M+K}e^{-t\sqrt{L}}\left(\mathbf{1}_{T_l^j}(\cdot,t)(t^2L)^K e^{-t\sqrt{L}}f(\cdot)\right)(x)\frac{dt}{t},$$

we show that, for some constant $C > 0$, we have the following $(w, q_1, p, \varepsilon_0, M)$-representation of f:

$$f = C\sum_{j\in\mathbb{N}, l\in\mathbb{Z}} \lambda_l^j \mathfrak{m}_l^j.$$

To that end, we have to show the following:

(1) $\{\lambda_l^j\} \in \ell^p$,

(2) for all $j \in \mathbb{N}$ and $l \in \mathbb{Z}$, there exists a constant $C_0 > 0$ such that $C_0^{-1}\mathfrak{m}_l^j$ is a $(w, q_1, p, \varepsilon_0, M)$−molecule,

(3) $f = C\sum_{j\in\mathbb{N},l\in\mathbb{Z}} \lambda_l^j \mathfrak{m}_l^j$ in $L^{q_1}(w)$.

Statement (1) follows from the definition of the cubes Q_l^j and the sets O_l^* and O_l, and from the fact that $\|\mathcal{S}_{K,\mathrm{P}}f\|_{L^p(w)} < \infty$. Indeed, proceeding as in (4.52)

$$\sum_{j\in\mathbb{N},l\in\mathbb{Z}} |\lambda_l^j|^p = \sum_{j\in\mathbb{N},l\in\mathbb{Z}} 2^{pl}w(Q_l^j) \leq \sum_{l\in\mathbb{Z}} 2^{pl}w(O_l^*) \lesssim \sum_{l\in\mathbb{Z}} 2^{pl}w(O_l) \lesssim \|\mathcal{S}_{K,\mathrm{P}}f\|_{L^p(w)}^p < \infty. \tag{4.56}$$

The proofs of (2) and (3) are similar to those of Proposition 4.40, part (a), so we shall skip some details. In order to show (2), fix $j \in \mathbb{N}$, $l \in \mathbb{Z}$ and $0 \leq k \leq M$, $k \in \mathbb{N}$. We need to compute the following norms, for every $i \geq 1$,

$$\left\|\left((\ell(Q_l^j)^2L)^{-k}\mathfrak{m}_l^j\right)\mathbf{1}_{C_i(Q_l^j)}\right\|_{L^{q_1}(w)}.$$

For $i = 1$, consider the operator $\mathcal{T}_t^* := (t^2L^*)^{2M+K-k}e^{-t\sqrt{L^*}}$, and define the operator Q_L acting over functions h defined in $\mathbb{R}^n$ by

$$Q_L h(x,t) := \mathcal{T}_t^* h(x), \quad \text{for all} \quad (x,t) \in \mathbb{R}_+^{n+1}.$$

Applying the subordination formula (1.36), we have that, for every $\widetilde{K} \in \mathbb{N}$ and any operator L,

$$\begin{aligned}\left(\int_0^\infty \left|(t^2L)^{\widetilde{K}}e^{-t\sqrt{L}}f(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}} &\lesssim \int_0^\infty e^{-u}u^{\frac{1}{2}}\left(\int_0^\infty \left|(t^2L)^{\widetilde{K}}e^{-\frac{t^2}{4u}L}f(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\frac{du}{u}\\ &\lesssim \int_0^\infty e^{-u}u^{\widetilde{K}+\frac{1}{2}}\frac{du}{u}\left(\int_0^\infty \left|(t^2L)^{\widetilde{K}}e^{-t^2L}f(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}} \lesssim \left(\int_0^\infty \left|(t^2L)^{\widetilde{K}}e^{-t^2L}f(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}.\end{aligned} \tag{4.57}$$

Therefore, for every $q_1' \in \mathcal{W}_{w^{1-q_1'}}(p_-(L^*), p_+(L^*))$ it follows from [11] that

$$\begin{aligned}\|Q_L h\|_{L_{\mathbb{H}}^{q_1'}(w^{1-q_1'})} &= \|\,\|Q_L h\|_{\mathbb{H}}\|_{L^{q_1'}(w^{1-q_1'})} = \left(\int_{\mathbb{R}^n}\left(\int_0^\infty \left|(t^2L^*)^{2M+K-k}e^{-t\sqrt{L^*}}h(x)\right|^2\frac{dt}{t}\right)^{\frac{q_1'}{2}} w^{1-q_1'}(x)dx\right)^{\frac{1}{q_1'}}\\ &\lesssim \left(\int_{\mathbb{R}^n}\left(\int_0^\infty \left|(t^2L^*)^{2M+K-k}e^{-t^2L^*}h(x)\right|^2\frac{dt}{t}\right)^{\frac{q_1'}{2}} w^{1-q_1'}(x)dx\right)^{\frac{1}{q_1'}} \lesssim \|h\|_{L^{q_1'}(w^{1-q_1'})}.\end{aligned}$$

Consequently, Q_L is bounded from $L^{q_1'}(w^{1-q_1'})$ to $L_{\mathbb{H}}^{q_1'}(w^{1-q_1'})$. Thus, proceeding as in Remark 3.49, we have that its adjoint Q_L^* has a bounded extension from $L_{\mathbb{H}}^{q_1}(w)$ to $L^{q_1}(w)$, for all $q_1 \in \mathcal{W}_w(p_-(L), p_+(L))$, where

$$Q_L^* f(x) = \int_0^\infty (t^2L)^{2M+K-k}e^{-t\sqrt{L}}f(x,t)\frac{dt}{t},$$

recalling that, abusing notation, L denotes the infinitesimal generator of e^{-tL} on $L^{q_1}(w)$ (see [11, Remark 3.5]).

Next, considering $f^j_{l,K}(x,t) := \mathbf{1}_{T^j_l}(x,t)(t^2L)^K e^{-t\sqrt{L}} f(x)$ and $\widetilde{g}(x,t) := t^{2k} f^j_{l,K}(x,t)$, and proceeding as in (4.50), we have

$$
\begin{aligned}
&\left\| \left(\left(\ell(Q^j_l)^2 L\right)^{-k} \mathfrak{m}^j_l \right) \mathbf{1}_{4Q^j_l} \right\|_{L^{q_1}(w)} \lesssim \frac{\ell(Q^j_l)^{-2k}}{\lambda^j_l} \left\| Q^*_L \widetilde{g} \right\|_{L^{q_1}(4Q^j_l, w)} \\
&= \frac{\ell(Q^j_l)^{-2k}}{\lambda^j_l} \sup_{\|h\|_{L^{q_1'}(w^{1-q_1'})}=1} \left| \int_{\mathbb{R}^n} Q^*_L \widetilde{g}(x) \cdot h(x) dx \right| \\
&= \frac{\ell(Q^j_l)^{-2k}}{\lambda^j_l} \sup_{\|h\|_{L^{q_1'}(w^{1-q_1'})}=1} \left| \int_{\mathbb{R}^n} \int_0^\infty \widetilde{g}(x,t) \cdot \mathcal{T}^*_t h(x) \frac{dt}{t} dx \right| \\
&\lesssim \frac{1}{\lambda^j_l} \left(\int_{cQ^j_l \cap E_{l+1}} |\mathcal{S}_{K,\mathrm{P}} f(x)|^{q_1} w(x) dx \right)^{\frac{1}{q_1}} \sup_{\|h\|_{L^{q_1'}(w^{1-q_1'})}=1} \left(\int_{cQ^j_l \cap E_{l+1}} \left(\iint_{\Gamma(x)} \left| \mathcal{T}^*_t h(y) \right|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{q_1'}{2}} w(x)^{1-q_1'} dx \right)^{\frac{1}{q_1'}} \\
&\lesssim w(Q^j_l)^{\frac{1}{q_1} - \frac{1}{p}}.
\end{aligned}
$$

The last inequality follows from the fact that the conical square function defined by $\mathcal{T}^*_t$ is bounded on $L^{q_1'}(w^{1-q_1'})$ (see Theorem 3.2), since $q_1 \in \mathcal{W}_w(p_-(L), p_+(L))$ implies $q_1' \in \mathcal{W}_{w^{1-q_1'}}(p_-(L^*), p_+(L^*))$ and also from the fact that $\mathcal{S}_{K,\mathrm{P}} f(x) \leq 2^{l+1}$ for all $x \in E_{l+1}$.

For $i \geq 2$, take $\max\{2, q_1\} < q_0 < p_+(L)$ close enough to $p_+(L)$ such that $w \in RH_{\left(\frac{q_0}{q_1}\right)'}$. Then, since $\{(t\sqrt{L})^{2\widetilde{K}} e^{-t\sqrt{L}}\}_{t>0} \in \mathcal{F}_{\widetilde{K}+\frac{1}{2}}(L^2 \to L^{q_0})$, for every $\widetilde{K} \in \mathbb{N}$, taking $\frac{r_w}{p} < \frac{r}{p} < \frac{r_w}{p} + \frac{1}{n}$ close enough to r_w so that $M > \frac{n}{2}\left(\frac{r}{p} - \frac{1}{2}\right)$, recalling that $0 \leq k \leq M$, and using (1.11), we have

$$
\begin{aligned}
&\left\| \left((\ell(Q^j_l)^2 L)^{-k} \mathfrak{m}^j_l \right) \mathbf{1}_{C_i(Q^j_l)} \right\|_{L^{q_1}(w)} \lesssim \frac{1}{\lambda^j_l} \ell(Q^j_l)^{-2k} w(2^{i+1} Q^j_l)^{\frac{1}{q_1}} |2^{i+1} Q^j_l|^{-\frac{1}{q_0}} \\
&\qquad \times \int_0^\infty t^{2k} \left(\int_{C_i(Q^j_l)} \left| (t^2 L)^{2M+K-k} e^{-t\sqrt{L}} \left(\mathbf{1}_{Q^j_l}(\cdot) f^j_{l,K}(\cdot, t) \right)(y) \right|^{q_0} dy \right)^{\frac{1}{q_0}} \frac{dt}{t} \\
&\lesssim 2^{-l} w(Q^j_l)^{-\frac{1}{p}} \ell(Q^j_l)^{-2k} w(2^{i+1} Q^j_l)^{\frac{1}{q_1}} |2^{i+1} Q^j_l|^{-\frac{1}{q_0}} \\
&\qquad \times \int_0^{5\sqrt{n}\ell(Q^j_l)} t^{2k - n\left(\frac{1}{2} - \frac{1}{q_0}\right)} \left(1 + \frac{c4^i \ell(Q^j_l)^2}{t^2} \right)^{-\left(2M+K-k+\frac{1}{2}+\frac{n}{2}\left(\frac{1}{2}-\frac{1}{q_0}\right)\right)} \left(\int_{Q^j_l} |f^j_{l,K}(y,t)|^2 dy \right)^{\frac{1}{2}} \frac{dt}{t} \\
&\lesssim 2^{-l} 2^{i\frac{rn}{p}} w(2^{i+1} Q^j_l)^{\frac{1}{q_1} - \frac{1}{p}} |2^{i+1} Q^j_l|^{-\frac{1}{q_0}} \left(\int_{cQ^j_l \cap E_{l+1}} |\mathcal{S}_{K,\mathrm{P}} f(x)|^2 dx \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^{5\sqrt{n}\ell(Q^j_l)} t^{-2n\left(\frac{1}{2} - \frac{1}{q_0}\right)} \left(1 + \frac{c4^i \ell(Q^j_l)^2}{t^2} \right)^{-\left(4M+2K-2k+1+n\left(\frac{1}{2}-\frac{1}{q_0}\right)\right)} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim 2^{-i\left(2M+2K+\frac{n}{2}+1-\frac{rn}{p}\right)} w(2^{i+1} Q^j_l)^{\frac{1}{q_1} - \frac{1}{p}} \\
&\leq 2^{-i\varepsilon_0} 2^{-i\left(\frac{r_w n}{p} - \frac{rn}{p} + 1\right)} w(2^{i+1} Q^j_l)^{\frac{1}{q_1} - \frac{1}{p}}.
\end{aligned}
$$

Therefore, it follows that $\|\mathfrak{m}^j_l\|_{mol,w} \leq C_0$ for some constant $C_0 > 0$ uniform in $j \in \mathbb{N}$ and $l \in \mathbb{Z}$.

Let us finally prove that $f = C\sum_{j\in\mathbb{N},l\in\mathbb{Z}} \lambda_l^j \mathfrak{m}_l^j$ in $L^{q_1}(w)$. We follow the same computations as in the proof of Proposition 4.40 part (a), we first see that by (4.57)

$$\Big\| \sum_{j\in\mathbb{N},l\in\mathbb{Z}} f_{l,m}^j \Big\|_{L^{q_1}_{\mathbb{H}}(w)} = \Big\| \sum_{j\in\mathbb{N},l\in\mathbb{Z}} |f_{l,m}^j| \Big\|_{L^{q_1}_{\mathbb{H}}(w)} \leq \Big\| \Big(\int_0^\infty |(t^2L)^K e^{-t\sqrt{L}} f|^2 \frac{dt}{t} \Big)^{\frac12} \Big\|_{L^{q_1}(w)}$$
$$\lesssim \Big\| \Big(\int_0^\infty |(t^2L)^K e^{-t^2 L} f|^2 \frac{dt}{t} \Big)^{\frac12} \Big\|_{L^{q_1}(w)} \lesssim \|f\|_{L^p(w)}.$$

This allows to obtain (4.54) where in this case $Q_L^M g(x) = (t^2L^*)^{2M+K} e^{-t\sqrt{L^*}} g(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$. Consequently, $f = C\sum_{j\in\mathbb{N},l\in\mathbb{Z}} \lambda_l^j \mathfrak{m}_l^j \in \mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w)$, and also, by (4.56)

$$\|f\|_{\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w)} \lesssim \left(\sum_{j\in\mathbb{N},l\in\mathbb{Z}} |\lambda_l^j|^p \right)^{\frac1p} \lesssim \|\mathcal{S}_{K,\mathrm{P}} f\|_{L^p(w)} = \|f\|_{\mathbb{H}^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w)}.$$

□

Proof of Proposition 4.55, part (b).

Given $w \in A_\infty$ and $q_1, q_2 \in \mathcal{W}_w(p_-(L), p_+(L))$, and $0 < p \leq 1$, from part (a), we have that

$$\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w) = \mathbb{H}^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w) \quad \text{and} \quad \mathbb{H}^p_{L,q_2,\varepsilon_0,M}(w) = \mathbb{H}^p_{\mathcal{S}_{K,\mathrm{P}},q_2}(w),$$

with equivalent norms. Hence we have the following isomorphisms

$$H^p_{L,q_1,\varepsilon_0,M}(w) \approx H^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w) \quad \text{and} \quad H^p_{L,q_2,\varepsilon_0,M}(w) \approx H^p_{\mathcal{S}_{K,\mathrm{P}},q_2}(w).$$

On the other hand, from Proposition 4.40, parts (a) and (b), we have that

$$H^p_{L,q_1,\varepsilon_0,M}(w) \approx H^p_{\mathcal{S}_{K,\mathrm{H}},q_1}(w) \approx H^p_{\mathcal{S}_{K,\mathrm{H}},q_2}(w) \approx H^p_{L,q_2,\varepsilon_0,M}(w).$$

Therefore we conclude that the spaces $H^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w)$ and $H^p_{\mathcal{S}_{K,\mathrm{P}},q_2}(w)$ are isomorphic. □

Proof of Proposition 4.55, part (c).

For $f \in \mathbb{H}^p_{\mathcal{G}_{K-1,\mathrm{P}},q_1}(w)$, applying Theorem 3.23, part (b), and the fact that $\mathrm{G}_{K-1,\mathrm{P}} f(x) \leq \mathcal{G}_{K-1,\mathrm{P}} f(x)$ for every $x \in \mathbb{R}^n$ and for every $K \in \mathbb{N}$, we conclude

$$\|\mathcal{S}_{K,\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathrm{G}_{K-1,\mathrm{P}} f\|_{L^p(w)} \leq \|\mathcal{G}_{K-1,\mathrm{P}} f\|_{L^p(w)}.$$

This, together with Proposition 4.55, part (a), implies

$$\mathbb{H}^p_{\mathcal{G}_{K-1,\mathrm{P}},q_1}(w) \subset \mathbb{H}^p_{\mathrm{G}_{K-1,\mathrm{P}},q_1}(w) \subset \mathbb{H}^p_{\mathcal{S}_{K,\mathrm{P}},q_1}(w) = \mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w).$$

To complete the proof, take $f \in \mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w)$. Then, since in particular $f \in L^{q_1}(w)$, and by Proposition 4.22, we have that

$$\|\mathcal{G}_{K-1,\mathrm{P}} f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w)}.$$

Then, we conclude that, $\mathbb{H}^p_{L,q_1,\varepsilon_0,M}(w) \subset \mathbb{H}^p_{\mathcal{G}_{K-1,\mathrm{P}},q_1}(w)$, which finishes the proof. □

4.3.3 Characterization of the weighted Hardy spaces associated with $\mathcal{N}_{\mathrm{H}}$ and $\mathcal{N}_{\mathrm{P}}$

We obtain Theorem 4.20 from the following proposition.

Proposition 4.58. *Let $w \in A_\infty$, $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \leq 1$, $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{2}\right)$, and $\varepsilon_0 = 2M + 2 + \frac{n}{2} - \frac{r_w n}{p}$, there hold*

(a) $\mathbb{H}^p_{\mathcal{N}_{\mathrm{H}},q}(w) = \mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w) = \mathbb{H}^p_{L,q,\varepsilon_0,M}(w)$, *with equivalent norms.*

(b) $\mathbb{H}^p_{\mathcal{N}_{\mathrm{P}},q}(w) = \mathbb{H}^p_{\mathcal{G}_{\mathrm{P}},q}(w) = \mathbb{H}^p_{L,q,\varepsilon_0,M}(w)$, *with equivalent norms.*

Proof. Fix $w \in A_\infty$, $q \in \mathcal{W}_w(p_-(L), p_+(L))$, $0 < p \leq 1$, $M \in \mathbb{N}$ such that $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{2}\right)$, and $\varepsilon_0 = 2M + 2 + \frac{n}{2} - \frac{r_w n}{p}$.

In order to prove part (a), note that for $f \in \mathbb{H}^p_{L,q,\varepsilon_0,M}(w)$, we have, in particular that $f \in L^q(w)$. Besides, by Proposition 4.31, part (b)

$$\|f\|_{\mathbb{H}^p_{\mathcal{N}_{\mathrm{H}},q}(w)} = \|\mathcal{N}_{\mathrm{H}} f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q,\varepsilon_0,M}(w)}.$$

Therefore, $f \in \mathbb{H}^p_{\mathcal{N}_{\mathrm{H}},q}(w)$.

Take now $f \in \mathbb{H}^p_{\mathcal{N}_{\mathrm{H}},q}(w)$. Theorems 3.23, part (a), and 3.29, part (b), and Remark 3.45 imply

$$\|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)} \lesssim \|\mathcal{N}_{\mathrm{H}} f\|_{L^p(w)}.$$

Then, $f \in \mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$, which implies that, by Proposition 4.40, part (a), $f \in \mathbb{H}^p_{L,q,\varepsilon_0,M}(w)$ and

$$\|f\|_{\mathbb{H}^p_{L,q,\varepsilon_0,M}(w)} \lesssim \|f\|_{\mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)} \lesssim \|f\|_{\mathbb{H}^p_{\mathcal{N}_{\mathrm{H}},q}(w)}.$$

As for proving part (b), take $f \in \mathbb{H}^p_{L,q,\varepsilon_0,M}(w)$ and apply Proposition 4.31, part (b), to obtain

$$\|f\|_{\mathbb{H}^p_{\mathcal{N}_{\mathrm{P}},q}(w)} = \|\mathcal{N}_{\mathrm{P}} f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q,\varepsilon_0,M}(w)}.$$

Hence $f \in \mathbb{H}^p_{\mathcal{N}_{\mathrm{P}},q}(w)$.

Finally, notice that for $f \in \mathbb{H}^p_{\mathcal{N}_{\mathrm{P}},q}(w)$ Theorem 3.29, part (a), and Remark 3.45 imply that

$$\|\mathcal{G}_{\mathrm{P}} f\|_{L^p(w)} \lesssim \|\mathcal{N}_{\mathrm{P}} f\|_{L^p(w)}.$$

Therefore, $f \in \mathbb{H}^p_{\mathcal{G}_{\mathrm{P}},q}(w)$. Then, applying Proposition 4.55, part (c), we conclude that

$$\|f\|_{\mathbb{H}^p_{L,q,\varepsilon_0,M}(w)} \lesssim \|f\|_{\mathbb{H}^p_{\mathcal{G}_{\mathrm{P}},q}(w)} \lesssim \|f\|_{\mathbb{H}^p_{\mathcal{N}_{\mathrm{P}},q}(w)}.$$

and $f \in \mathbb{H}^p_{L,q,\varepsilon_0,M}(w)$. □

4.4 Characterization of $H^p_{\mathcal{T}}(w)$, $p \in \mathcal{W}_w(p_-(L), p_+(L))$

For $\mathcal{T}$ being any square function in (1.26)-(1.31) or a non-tangential maximal function in (1.32), we have that the Hardy spaces $H^p_{\mathcal{T}}(w)$ are isomorphic to the $L^p(w)$ spaces, for an appropriate range of p.

Theorem 4.59. *Given $w \in A_\infty$, if $\mathcal{T}$ is any of the square functions in* (1.26)-(1.31) *or a non-tangential maximal function in* (1.32)*, then, for all $p \in \mathcal{W}_w(p_-(L), p_+(L))$, the spaces $H^p_{\mathcal{T}}(w)$ and $L^p(w)$ are isomorphic, with equivalent norms.*

Proof. For $w \in A_\infty$ and $p, q \in \mathcal{W}_w(p_-(L), p_+(L))$, we claim that $L^q(w) \cap L^p(w) = \mathbb{H}^p_{\mathcal{T},q}(w)$ with

$$\|f\|_{\mathbb{H}^p_{\mathcal{T},q}(w)} \approx \|f\|_{L^p(w)}, \tag{4.60}$$

where $\mathcal{T}$ is any function defined in (1.26)-(1.31) or (1.32). Then, taking the closure we would conclude the desired isomorphism

$$H^p_{\mathcal{T},q}(w) \approx L^p(w), \text{ for all } p, q \in \mathcal{W}_w(p_-(L), p_+(L)),$$

with constants independent of q, so we can drop the dependence on q and just write

$$H^p_{\mathcal{T}}(w) \approx L^p(w), \text{ for all } p \in \mathcal{W}_w(p_-(L), p_+(L)).$$

Let us prove our claim. If $f \in L^p(w) \cap L^q(w)$, since $\|\mathcal{T} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} < \infty$, (see Theorems 3.1, 3.2, and 3.25), then $f \in \mathbb{H}^p_{\mathcal{T},q}(w)$.

In order to show the converse inclusion, let us first consider the particular case of $\mathcal{T} \equiv \mathcal{S}_{m,\mathrm{P}}$, for $m \in \mathbb{N}$. Then, take $f \in \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{P}},q}(w)$, and consider the operator Q_L defined by

$$Q_L h(x,t) := \mathcal{T}_t^* h(x), \quad \text{for all} \quad (x,t) \in \mathbb{R}^{n+1}_+,$$

where $\mathcal{T}_t^* := (t^2L^*)^m e^{-t\sqrt{L^*}}$. We have that this operator is bounded from $L^{p'}(w^{1-p'})$ to $T^{p'}(w^{1-p'})$, for all $p' \in \mathcal{W}_{w^{1-p'}}(p_-(L^*), p_+(L^*))$. Indeed, by Theorem 3.2, we have that, for every $h \in L^{p'}(w^{1-p'})$,

$$\|Q_L h\|_{T^{p'}(w^{1-p'})} = \left(\int_{\mathbb{R}^n} \left(\iint_{\Gamma(x)} |(t^2L^*)^m e^{-t\sqrt{L^*}} h(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p'}{2}} w^{1-p'}(x) dx \right)^{\frac{1}{p'}} \lesssim \|h\|_{L^{p'}(w^{1-p'})}.$$

Hence, proceeding as in the proof of Proposition 3.46, we conclude that its adjoint Q_L^*, has a bounded extension from $T^p(w)$ to $L^p(w)$, for all $p \in \mathcal{W}_w(p_-(L), p_+(L))$, where

$$Q_L^* h(x) = \int_0^\infty (t^2L)^m e^{-t\sqrt{L}} h(x,t) \frac{dt}{t},$$

and abusing notation, L denotes the infinitesimal generator of e^{-tL} on $L^p(w)$ (see [11, Remark 3.5]). Next, consider the following Calderón reproducing formula of f

$$f(x) = C_m \int_0^\infty \left((t^2L)^m e^{-t\sqrt{L}} \right)^2 f(x) \frac{dt}{t},$$

where the equality is in $L^q(w)$ (see Remark 3.49). Now, since for $f \in \mathbb{H}^p_{\mathcal{S}_{m,\mathrm{P}},q}(w)$, we have that $f \in L^q(w)$ and that $\widetilde{f}(x,t) := (t^2L)^m e^{-t\sqrt{L}} f(x) \in T^p(w)$ (because $\|\widetilde{f}\|_{T^p(w)} = \|\mathcal{S}_{m,K} f\|_{L^p(w)} < \infty$), we get, for $g \in L^{p'}(w) \cap L^{q'}(w)$,

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} f(y) \bar{g}(y) w(y)\, dy \right| &= C_m \left| \int_{\mathbb{R}^n} Q_L^* \widetilde{f}(y) \bar{g}(y) w(y)\, dy \right| \\
&= C_m \left| \int_{\mathbb{R}^n} \int_0^\infty (t^2L)^m e^{-t\sqrt{L}} \widetilde{f}(y,t) \frac{dt}{t} \bar{g}(y) w(y)\, dy \right| \\
&= C_m \left| \int_{\mathbb{R}^n} \int_0^\infty \widetilde{f}(y,t) (t^2L^*)^m e^{-t\sqrt{L^*}} (\bar{g}w)(y) \frac{dt}{t}\, dy \right| \\
&\lesssim \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)} \|Q_L(\bar{g}w)\|_{T^{p'}(w^{1-p'})} \\
&\lesssim \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)} \|gw\|_{L^{p'}(w^{1-p'})} = \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)} \|g\|_{L^{p'}(w)}.
\end{aligned}$$

Then, taking the supremum over all $g \in L^{p'}(w) \cap L^{q'}(w)$ such that $\|g\|_{L^{p'}(w)} = 1$ (note that $L^{q'}(w) \cap L^{p'}(w)$ is dense in $L^{p'}(w)$), we obtain that

$$\|f\|_{L^p(w)} \lesssim \|\mathcal{S}_{m,\mathrm{P}} f\|_{L^p(w)}. \tag{4.61}$$

Therefore, we have that, for all $m \in \mathbb{N}$, $\mathbb{H}^p_{\mathcal{S}_{m,\mathrm{P}},q}(w) = L^p(w) \cap L^q(w)$, with equivalent norms.

Finally, by (3.13), (4.61), Theorems 3.3 and 3.4, Remark 3.22, and Theorems 3.23 and 3.29, we obtain that $\mathbb{H}^p_{\mathcal{T},q}(w) = L^p(w) \cap L^q(w)$, for all $p,\ q \in \mathcal{W}_w(p_-(L), p_+(L))$ and for $\mathcal{T}$ being any square function in (1.26)-(1.31), or a non-tangential maximal function in (1.32), with equivalent norms. □

Remark 4.62. *As we explain in the proof we have obtained the isomorphism $H^p_{\mathcal{T},q}(w) \approx L^p(w)$ for all $p, q \in \mathcal{W}_w(p_-(L), p_+(L))$. In particular, this implies that*

$$H^p_{\mathcal{T},q_1}(w) \approx H^p_{\mathcal{T},q_2}(w), \text{ for all } p, q_1, q_2 \in \mathcal{W}_w(p_-(L), p_+(L)).$$

for $\mathcal{T}$ being any function in (1.26)-(1.31)*, or a non-tangential maximal function in* (1.32)*.*

4.5 Characterization of the weighted Hardy space associated with the Riesz transform

In order to characterize the weighted Hardy space associated with the Riesz transform we proceed as in [56], where the unweighted case was consider. First of all, we need to prove the following weighted versions of [56, Propositions 5.32 and 5.34].

Proposition 4.63. *Given $w \in A_\infty$ and $q \in \mathcal{W}_w(q_-(L), q_+(L))$, for all p satisfying* $\max\left\{r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)}\right\} < p < \frac{p_+(L)}{s_w}$ *and $f \in \mathbb{H}^p_{\nabla L^{-1/2},q}(w)$, we have that*

$$\|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)} \lesssim \|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)}. \tag{4.64}$$

In particular, we conclude that, for all $\max\left\{r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)}\right\} < p < \frac{p_+(L)}{s_w}$, $\mathbb{H}^p_{\nabla L^{-1/2},q}(w) \subset \mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$.

Proposition 4.65. *Given $w \in A_\infty$ and $q \in \mathcal{W}_w(q_-(L), q_+(L))$, for all* $0 < p < \frac{q_+(L)}{s_w}$ *and $f \in \mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$, we have that*

$$\|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)}.$$

In particular, we conclude that, for all $0 < p < \frac{q_+(L)}{s_w}$, $\mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w) \subset \mathbb{H}^p_{\nabla L^{-1/2},q}(w)$.

From these results we obtain at once that the weighted Hardy space defined by the Riesz transform is isomorphic to the weighted Hardy space defined by the conical square function $\mathcal{S}_{\mathrm{H}}$.

Theorem 4.66. *Given $w \in A_\infty$ such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. For all* $\max\left\{r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)}\right\} < p < \frac{q_+(L)}{s_w}$ *and $q \in \mathcal{W}_w(q_-(L), q_+(L))$, we have the following isomorphism*

$$H^p_{\mathcal{S}_{\mathrm{H}},q}(w) \approx H^p_{\nabla L^{-1/2},q}(w).$$

Remark 4.67. *We observe that in view of Theorem 4.59, the dependence on q in the above isomorphism can be omitted when $p \in \mathcal{W}_w(q_-(L), q_+(L)) \subset \mathcal{W}_w(p_-(L), p_+(L))$.*

In order to prove Propositions 4.63 and 4.65, we need the following propositions. In the first one, we study the action of the Riesz transform over molecules.

Proposition 4.68. *For every $w \in A_\infty$, $q \in \mathcal{W}_w(q_-(L), q_+(L))$, $0 < p \leq 1$, $\varepsilon > 0$, $M \in \mathbb{N}$ such that* $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{p_-(L)}\right)$, *and $\mathfrak{m}$ a $(w, q, p, \varepsilon, M)$-molecule, there hold*

(*a*) $\|\nabla L^{-\frac{1}{2}} \mathfrak{m}\|_{L^p(w)} \leq C$.

(*b*) *For all* $f \in \mathbb{H}^p_{L,q,\varepsilon,M}(w)$, $\|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{L,q,\varepsilon,M}(w)}$.

In the second we compare the norm of the following conical square function:

$$\widetilde{\mathcal{S}}f(x) := \left(\iint_{\Gamma(x)} |t\sqrt{L}e^{-t^2L}f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

with the norm of $\mathcal{S}_{\mathrm{H}}$ in $L^p(w)$, in some range of p.

Proposition 4.69. *Given* $w \in A_\infty$ *and* $f \in L^2(\mathbb{R}^n)$, *there hold*

(*a*) $\|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)} \lesssim \|\widetilde{\mathcal{S}}f\|_{L^p(w)}$, *for all* $p \in \mathcal{W}_w(0, p_+(L)^{1/2,*})$;

(*b*) $\|\widetilde{\mathcal{S}}f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)}$, *for all* $p \in \mathcal{W}_w(0, p_+(L)^*)$.

In particular

$$\|\widetilde{\mathcal{S}}f\|_{L^p(w)} \approx \|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)}, \text{ for all } p \in \mathcal{W}_w(0, p_+(L)^*).$$

Proof of Proposition 4.68.

Assuming part (*a*) the proof of part (*b*) follows as the proof of Proposition 4.22, part (*b*), but using Theorem 1.34 instead of Theorems 3.1 and 3.2.

We next prove part (*a*). Fix w, p, q, ε, and M as in the statement of the proposition. Note that since $w \in A_{\frac{q}{q_-(L)}} \cap RH_{\left(\frac{q_+(L)}{q}\right)'}$ and $M > \frac{n}{2}\left(\frac{r_w}{p} - \frac{1}{q_-(L)}\right)$ (recall that $p_-(L) = q_-(L)$), we can take $r_0 > r_w$, p_0, and q_0, $q_-(L) < p_0 < q < q_0 < q_+(L)$, close enough to r_w, $q_-(L)$, and $q_+(L)$, respectively, so that $w \in A_{\frac{q}{p_0}} \cap RH_{\left(\frac{q_0}{q}\right)'}$ and

$$M > \frac{n}{2}\left(\frac{r_0}{p} - \frac{1}{p_0}\right). \tag{4.70}$$

Besides, take $\mathfrak{m}$ a $(w, q, p, \varepsilon, M)$−molecule and $Q \subset \mathbb{R}^n$ one of its associated cubes, with sidelength $\ell(Q)$, and consider

$$B_Q := \left(I - e^{-\ell(Q)^2L}\right)^M \quad \text{and} \quad A_Q := I - B_Q.$$

Recalling the notation given in (4.6), considering $\widetilde{A}^k_Q := \left(k\ell(Q)^2L\right)^M e^{-k\ell(Q)^2L}$, we can write

$$\begin{aligned}\nabla L^{-\frac{1}{2}}\mathfrak{m} = \nabla L^{-\frac{1}{2}}B_Q\mathfrak{m} + \nabla L^{-\frac{1}{2}}A_Q\mathfrak{m} &= \nabla L^{-\frac{1}{2}}B_Q\mathfrak{m} + \sum_{k=1}^{M} C_{k,M}\nabla L^{-\frac{1}{2}}\widetilde{A}^k_Q\widetilde{\mathfrak{m}} \\ &= \sum_{i\geq 1}\left(\nabla L^{-\frac{1}{2}}B_Q\mathfrak{m}_i + \sum_{k=1}^{M} C_{k,M}\nabla L^{-\frac{1}{2}}\widetilde{A}^k_Q\widetilde{\mathfrak{m}}_i\right), \end{aligned}\tag{4.71}$$

where $\widetilde{\mathfrak{m}} := (\ell(Q)^2L)^{-M}\mathfrak{m}$ and for any function f, we denote $f_i := f\mathbf{1}_{C_i(Q)}$, for all $i \geq 1$. Then, we have

$$\|\nabla L^{-\frac{1}{2}}B_Q\mathfrak{m}_i\|^p_{L^p(w)} \lesssim \sum_{j\geq 1}\|\mathbf{1}_{C_j(Q_i)}\nabla L^{-\frac{1}{2}}(B_Q\mathfrak{m}_i)\|^p_{L^p(w)} =: \sum_{j\geq 1} I_{ij}. \tag{4.72}$$

Thus, for $j = 1$, applying Hölder's inequality, the boundedness of $\nabla L^{-\frac{1}{2}}$ and B_Q on $L^q(w)$ (see Theorem 1.34 and Proposition 1.41), by (1.11) and (4.8), we obtain

$$I_{i1} \lesssim w(2^{i+1}Q)^{1-\frac{p}{q}}\left(\int_{4Q_i}\left|\nabla L^{-\frac{1}{2}}B_Q\mathfrak{m}_i(x)\right|^q w(x)dx\right)^{\frac{p}{q}} \lesssim \|\mathfrak{m}_i\|^p_{L^q(w)}w(2^{i+1}Q)^{1-\frac{p}{q}} \leq 2^{-ip\varepsilon}. \tag{4.73}$$

As for $j \geq 2$, denoting $\mathcal{T}_t := t\nabla_y e^{-t^2 L}$, using (1.33) and splitting the integral in t, we obtain

$$I_{ij} \leq \int_{C_j(Q_i)} \left| \int_0^{\ell(Q)} \mathcal{T}_t B_Q \mathfrak{m}_i(x) \frac{dt}{t} \right|^p w(x)dx + \int_{C_j(Q_i)} \left| \int_{\ell(Q)}^{\infty} \mathcal{T}_t B_Q \mathfrak{m}_i(x) \frac{dt}{t} \right|^p w(x)dx =: I_{ij}^1 + I_{ij}^2. \tag{4.74}$$

In order to estimate I_{ij}^1, we apply twice Hölder's inequality, the fact that $w \in RH_{\left(\frac{q_0}{q}\right)'}$, and Mikowski's integral inequality. Besides, we expand the binomial and apply the fact that $\{\mathcal{T}_t\}_{t>0} \in \mathcal{F}_\infty(L^{p_0} \to L^{q_0})$ and $\{e^{-t^2 L}\}_{t>0} \in \mathcal{F}_\infty(L^{p_0} \to L^{p_0})$ (see Section 1.3.1). Also, we apply Lemma 1.35, Lemma 4.9, and (1.10). Then,

$$\begin{aligned}
I_{ij}^1 &\lesssim w(2^{j+1}Q_i)^{1-\frac{p}{q}} \left(\int_{C_j(Q_i)} \left| \int_0^{\ell(Q)} \mathcal{T}_t B_Q \mathfrak{m}_i(x) \frac{dt}{t} \right|^q w(x)dx \right)^{\frac{p}{q}} \\
&\lesssim w(2^{j+1}Q_i)\, |2^{j+1}Q_i|^{-\frac{p}{q_0}} \left(\int_0^{\ell(Q)} \left(\int_{C_j(Q_i)} \left| \mathcal{T}_t B_Q \mathfrak{m}_i(x) \right|^{q_0} dx \right)^{\frac{1}{q_0}} \frac{dt}{t} \right)^p . \\
&\lesssim w(2^{j+1}Q_i)\, |2^{j+1}Q_i|^{-\frac{p}{q_0}} \left(\int_0^{\ell(Q)} \left(\int_{C_j(Q_i)} \left| t\nabla_y e^{-t^2 L} \mathfrak{m}_i(x) \right|^{q_0} dx \right)^{\frac{1}{q_0}} \frac{dt}{t} \right. \\
&\qquad \left. + \sum_{k=1}^{M} C_{k,M} \int_0^{\ell(Q)} \ell(Q)^{-1} \left(\int_{C_j(Q_i)} \left| \sqrt{k}\ell(Q)\nabla_y e^{-k\ell(Q)^2 L} e^{-t^2 L} \mathfrak{m}_i(x) \right|^{q_0} dx \right)^{\frac{1}{q_0}} dt \right)^p \\
&\lesssim \|\mathfrak{m}_i\|_{L^{p_0}(\mathbb{R}^n)}^p w(2^{j+1}Q_i) |2^{j+1}Q_i|^{-\frac{p}{q_0}} \left(\int_0^{\ell(Q)} \left(e^{-c\frac{4^{i+j}\ell(Q)^2}{t^2}} t^{-\left(\frac{n}{p_0}-\frac{n}{q_0}\right)} + \frac{te^{-c4^{i+j}}}{\ell(Q)} \ell(Q)^{-\left(\frac{n}{p_0}-\frac{n}{q_0}\right)} \right) \frac{dt}{t} \right)^p \\
&\lesssim e^{-c4^{i+j}}.
\end{aligned} \tag{4.75}$$

As for I_{ij}^2, we proceed as before but also changing the variable t into $\sqrt{M+1}t =: C_M t$ and considering $B_{Q,t} := \left(e^{-t^2 L} - e^{-(t^2+\ell(Q)^2)L} \right)^M$. Next, we apply Lemma 1.35 using that $\{\mathcal{T}_t\}_{t>0} \in \mathcal{F}_\infty(L^{p_0} \to L^{q_0})$ and the $L^{p_0}(\mathbb{R}^n) - L^{p_0}(\mathbb{R}^n)$ off-diagonal estimates satisfied by $\{B_{Q,t}\}_{t>0}$ (see Section 1.3.1 and Proposition 1.37).Besides, note that $\ell(Q) \leq tC_M$. Then, by Lemma 4.9 and changing the variable t into $\frac{2^{j+i}\ell(Q)}{t}$, we get

$$\begin{aligned}
I_{ij}^2 &\lesssim w(2^{j+1}Q_i)|2^{j+1}Q_i|^{-\frac{p}{q_0}} \left(\int_{\frac{\ell(Q)}{C_M}}^{\infty} \left(\int_{C_j(Q_i)} \left| \mathcal{T}_t B_{Q,t} \mathfrak{m}_i(x) \right|^{q_0} dx \right)^{\frac{1}{q_0}} \frac{dt}{t} \right)^p \\
&\lesssim w(2^{j+1}Q_i)|2^{j+1}Q_i|^{-\frac{p}{q_0}} \|\mathfrak{m}_i\|_{L^{p_0}(\mathbb{R}^n)}^p \left(\int_{\frac{\ell(Q)}{C_M}}^{\infty} \left(\frac{\ell(Q)^2}{t^2} \right)^M t^{-n\left(\frac{1}{p_0}-\frac{1}{q_0}\right)} e^{-c\frac{4^{i+j}\ell(Q)^2}{t^2}} \frac{dt}{t} \right)^p \\
&\lesssim 2^{-ip(2M+\varepsilon)} 2^{-jp\left(2M+\frac{n}{p_0}-\frac{r_0 n}{p}\right)}.
\end{aligned}$$

Hence, by this, (4.72), (4.73), (4.74), (4.75), and by (1.11), we have

$$\sum_{i\geq 1} \|\nabla L^{-1/2} B_Q \mathfrak{m}_i\|_{L^p(w)}^p \leq C. \tag{4.76}$$

Now, proceeding as in the estimate of I_{i1}, since the Riesz transform and $\widetilde{A}_Q^k$ are bounded on $L^q(w)$ (see Theorem 1.34 and Proposition 1.41), and by (4.8), we get

$$\|\mathbf{1}_{4Q_i} \nabla L^{-\frac{1}{2}} \widetilde{A}_Q^k \widetilde{\mathfrak{m}}_i\|_{L^p(w)} \lesssim w(4Q_i)^{\frac{1}{p}-\frac{1}{q}} \|\widetilde{\mathfrak{m}}_i\|_{L^q(w)} \lesssim 2^{-i\varepsilon}. \tag{4.77}$$

Next, for $j \geq 2$, we use (1.33), and proceed as in the estimate of I_{ij},

$$\|\mathbf{1}_{C_j(Q_i)}\nabla L^{-\frac{1}{2}}\widetilde{A}_Q^k\widetilde{\mathfrak{m}}_i\|_{L^p(w)}^p \lesssim w(2^{j+1}Q_i)|2^{j+1}Q_i|^{-\frac{p}{q_0}}\left(\int_0^{\ell(Q)}\left(\int_{C_j(Q_i)}\left|\mathcal{T}_t\widetilde{A}_Q^k\widetilde{\mathfrak{m}}_i(x)\right|^{q_0}dx\right)^{\frac{1}{q_0}}\frac{dt}{t}\right.$$
$$\left.+\int_{\ell(Q)}^{\infty}\left(\int_{C_j(Q_i)}\left|\mathcal{T}_t\widetilde{A}_Q^k\widetilde{\mathfrak{m}}_i(x)\right|^{q_0}dx\right)^{\frac{1}{q_0}}\frac{dt}{t}\right)^p =: w(2^{j+1}Q_i)|2^{j+1}Q_i|^{-\frac{p}{q_0}}\left(II_{ij}^1+II_{ij}^2\right)^p.$$

We first estimate II_{ij}^1 acting similarly as in (4.75) when we dealt with I_{ij}^1. We apply Lemma 1.35 using that $k^{\frac{1}{2}}\ell(Q)\nabla_y\widetilde{A}_Q^k$ satisfies $L^{p_0}(\mathbb{R}^n) - L^{q_0}(\mathbb{R}^n)$ off-diagonal estimates and $\{e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^{p_0} \to L^{p_0})$. Then, by Lemma 4.9, we obtain

$$II_{ij}^1 = \frac{c_{k,M}}{\ell(Q)}\int_0^{\ell(Q)}\left(\int_{C_j(Q_i)}\left|k^{\frac{1}{2}}\ell(Q)\nabla_y\widetilde{A}_Q^k e^{-t^2L}\widetilde{\mathfrak{m}}_i(x)\right|^{q_0}dx\right)^{\frac{1}{q_0}}dt$$
$$\lesssim e^{-c4^{j+i}}\ell(Q)^{-n\left(\frac{1}{p_0}-\frac{1}{q_0}\right)}\|\widetilde{\mathfrak{m}}_i\|_{L^{p_0}(\mathbb{R}^n)} \lesssim w(Q_i)^{-\frac{1}{p}}\ell(Q)^{\frac{n}{q_0}}e^{-c4^{i+j}}. \tag{4.78}$$

Now consider $s_{Q,t}^k := k\ell(Q)^2 + t^2$. Then changing the variable t into $\sqrt{2}t$; and proceeding as in the estimate of I_{ij}^2 but applying this time Lemma 1.35 with the families $\{\mathcal{T}_t\}_{t>0} \in \mathcal{F}_\infty(L^{p_0} \to L^{q_0})$ and $\{(s_{Q,t}^k L)^M e^{-s_{Q,t}^k L}\}_{t>0}$ that satisfies $L^{p_0}(\mathbb{R}^n) - L^{p_0}(\mathbb{R}^n)$ off-diagonal estimates, we have that

$$II_{ij}^2 \lesssim \int_{\frac{\ell(Q)}{\sqrt{2}}}^{\infty}\left(\frac{\ell(Q)^2}{t^2}\right)^M\left(\int_{C_j(Q_i)}\left|\mathcal{T}_t\left(s_{Q,t}^k L\right)^M e^{-s_{Q,t}^k L}\widetilde{\mathfrak{m}}_i(x)\right|^{q_0}dx\right)^{\frac{1}{q_0}}\frac{dt}{t}$$
$$\lesssim \|\widetilde{\mathfrak{m}}_i\|_{L^{p_0}(\mathbb{R}^n)}\int_{\frac{\ell(Q)}{\sqrt{2}}}^{\infty}\left(\frac{\ell(Q)^2}{t^2}\right)^M t^{-n\left(\frac{1}{p_0}-\frac{1}{q_0}\right)}e^{-c\frac{4^{j+i}\ell(Q)^2}{t^2}}\frac{dt}{t} \lesssim 2^{-j\left(2M+n\left(\frac{1}{p_0}-\frac{1}{q_0}\right)\right)}\ell(Q)^{\frac{n}{q_0}}w(Q_i)^{-\frac{1}{p}}2^{-i\left(2M-\frac{n}{q_0}+\varepsilon\right)}.$$

Therefore, from this inequality and (4.78), we get

$$\|\mathbf{1}_{C_j(Q_i)}\nabla L^{-\frac{1}{2}}\widetilde{A}_Q^k\widetilde{\mathfrak{m}}_i\|_{L^p(w)}^p \lesssim e^{-c4^{i+j}} + 2^{-jp\left(2M+\frac{n}{p_0}-\frac{r_0 n}{p}\right)}2^{-ip(2M+\varepsilon)}.$$

This, (4.77), and (4.70) give us $\sum_{i\geq 1}\|\nabla L^{-\frac{1}{2}}\widetilde{A}_Q^k\widetilde{\mathfrak{m}}_i\|_{L^p(w)}^p \leq C$, which, together with (4.76) and in view of (4.71), allows us to conclude the proof. □

Proof of Proposition 4.69.

We first prove part (a). Note that since $2 < p_+(L)^{1/2,*}$, in view of Theorem 1.46, part (b), (or part (e) if $p_+(L)^{1/2,*} = \infty$), it is enough to prove it for $p = 2$ and all $w \in RH_{\left(\frac{p_+(L)^{1/2,*}}{2}\right)'}$. Assuming this, note that as in the estimate of term II when proving (4.16), given $w \in RH_{\left(\frac{p_+(L)^{1/2,*}}{2}\right)'}$, we can find q_0 and r, so that $2 < q_0 < p_+(L)$, $q_0/2 \leq r < \infty$, $w \in RH_{r'}$, and

$$2 + \frac{n}{2r} - \frac{n}{q_0} > 0. \tag{4.79}$$

After this observation we show the desired estimate. Using (1.20) and Minkowski's integral inequality, we obtain that

$$\mathcal{S}_{\mathrm{H}}f(x) \lesssim \left(\int_0^\infty \left(\int_0^t \left(\int_{B(x,t)} |sLe^{-s^2L}t^2\sqrt{L}e^{-t^2L}f(y)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{1}{2}}$$
$$+\left(\int_0^\infty \left(\int_t^\infty \left(\int_{B(x,t)} |sLe^{-s^2L}t^2\sqrt{L}e^{-t^2L}f(y)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{1}{2}} =: I+II.$$

In the case that $s<t$, we use the fact that $\{t^2Le^{-t^2L}\}_{t>0}, \{e^{-s^2L}\}_{s>0} \in \mathcal{F}_\infty(L^2 \to L^2)$, and apply Lemma 1.35 to get

$$I \leq \left(\int_0^\infty \left(\int_0^t \frac{s}{t}\left(\int_{B(x,t)} |e^{-s^2L}t^2Le^{-\frac{t^2}{2}L}t\sqrt{L}e^{-\frac{t^2}{2}L}f(y)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{1}{2}}$$
$$\lesssim \sum_{j\geq 1} e^{-c4^j}\left(\int_0^\infty \left(\int_0^t \frac{s}{t}\frac{ds}{s}\right)^2 \int_{B(x,2^{j+1}t)} |t\sqrt{L}e^{-\frac{t^2}{2}L}f(y)|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

Then, changing the variable t into $\sqrt{2}t$ and applying change of angles (Proposition 2.43), we conclude that

$$\|I\|_{L^2(w)} \lesssim \sum_{j\geq 1} e^{-c4^j}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2^{j+2}t)} |t\sqrt{L}e^{-t^2L}f(y)|^2\frac{dy\,dt}{t^{n+1}}w(x)dx\right)^{\frac{1}{2}} \lesssim \sum_{j\geq 1} e^{-c4^j}\|\widetilde{\mathcal{S}}f\|_{L^2(w)} \lesssim \|\widetilde{\mathcal{S}}f\|_{L^2(w)}.$$

As for the estimate of II, consider $\widetilde{f}(y,s) := s\sqrt{L}e^{-\frac{s^2}{2}L}f(y)$, apply the fact that $\{e^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^2)$ and Jensen's inequality. Besides, change the variable s into st, apply Minkowski's integral inequality, and then change the variable t into t/s. Hence, we have

$$II \lesssim \sum_{j\geq 1} e^{-c4^j}\left(\int_0^\infty\left(\int_t^\infty \frac{t^2}{s^2}\left(\int_{B(x,2^{j+1}t)} |s^2Le^{-\frac{s^2}{2}L}\widetilde{f}(y,s)|^2dy\right)^{\frac{1}{2}}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac{1}{2}}$$
$$\lesssim \sum_{j\geq 1} e^{-c4^j}\left(\int_0^\infty\left(\int_t^\infty \frac{t^2}{s^2}\left(\int_{B(x,2^{j+1}t)} |s^2Le^{-\frac{s^2}{2}L}\widetilde{f}(y,s)|^{q_0}\frac{dy}{t^n}\right)^{\frac{1}{q_0}}\frac{ds}{s}\right)^2\frac{dt}{t}\right)^{\frac{1}{2}}$$
$$\lesssim \sum_{j\geq 1} e^{-c4^j}\int_1^\infty s^{-2}\left(\int_0^\infty\left(\int_{B(x,2^{j+1}t)} |(st)^2Le^{-\frac{(st)^2}{2}L}\widetilde{f}(y,st)|^{q_0}\frac{dy}{t^n}\right)^{\frac{2}{q_0}}\frac{dt}{t}\right)^{\frac{1}{2}}\frac{ds}{s}$$
$$\lesssim \sum_{j\geq 1} e^{-c4^j}\int_1^\infty s^{-2+\frac{n}{q_0}}\left(\int_0^\infty\left(\int_{B(x,2^{j+1}t/s)} |t^2Le^{-\frac{t^2}{2}L}\widetilde{f}(y,t)|^{q_0}\frac{dy}{t^n}\right)^{\frac{2}{q_0}}\frac{dt}{t}\right)^{\frac{1}{2}}\frac{ds}{s}$$
$$=: \sum_{j\geq 1} e^{-c4^j}\int_1^\infty s^{-2+\frac{n}{q_0}}\mathcal{J}(x,s)\frac{ds}{s}.$$

In order to estimate the norm in $L^2(w)$ of the above integral, we first apply Minkowski's inequality, Proposition 2.61, and change the variable t into $\sqrt{2}t$. Next, we apply the fact that $\{t^2Le^{-t^2L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^{q_0})$, and recall that q_0 and r satisfy $2<q_0<p_+(L)$, $\frac{q_0}{2} \leq r$, $w \in RH_{r'}$, and (4.79). Finally, applying Proposition 2.43, we have

$$\left(\int_{\mathbb{R}^n}\left(\int_1^\infty s^{-2+\frac{n}{q_0}}\mathcal{J}(x,s)\frac{ds}{s}\right)^2 w(x)dx\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\lesssim \left(\int_1^\infty s^{-2-\frac{n}{2r}+\frac{n}{q_0}}\frac{ds}{s}\right)\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\int_{B(x,2^{j+2}t)}|t^2Le^{-t^2L}t\sqrt{L}e^{-t^2L}f(y)|^{q_0}\frac{dy}{t^n}\right)^{\frac{2}{q_0}}w(x)dx\frac{dt}{t}\right)^{\frac12}\\
&\lesssim \sum_{l\geq 1}e^{-c4^l}\left(\int_0^\infty\int_{\mathbb{R}^n}\int_{B(x,2^{j+l+3}t)}|t\sqrt{L}e^{-t^2L}f(y)|^2\frac{dy}{t^n}w(x)dx\frac{dt}{t}\right)^{\frac12}\\
&\lesssim \left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2^{j+1}t)}|t\sqrt{L}e^{-t^2L}f(y)|^2\frac{dy\,dt}{t^{n+1}}w(x)dx\right)^{\frac12}.
\end{aligned}
$$

Consequently, applying again Proposition 2.43, we get

$$
\|II\|_{L^2(w)} \lesssim \sum_{j\geq 1}e^{-c4^j}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,2^{j+1}t)}|t\sqrt{L}e^{-t^2L}f(y)|^2\frac{dy\,dt}{t^{n+1}}w(x)dx\right)^{\frac12}\lesssim \|\widetilde{\mathcal{S}}f\|_{L^2(w)},
$$

which, together with the estimate obtained for $\|I\|_{L^2(w)}$, gives us the desired inequality.

As for proving part (b), note that again it is enough to consider the case $p=2$ and $w\in RH_{\left(\frac{p_+(L)^*}{2}\right)'}$. In this case we find q_0 and r so that $2<q_0<p_+(L)$, $q_0/2\leq r<\infty$, $w\in RH_{r'}$, and

$$
1+\frac{n}{2r}-\frac{n}{q_0}>0. \tag{4.80}
$$

To this end, we proceed again as in the estimate of term II when proving (4.16), so we skip some details. For $n>p_+(L)$, note that we can take $\varepsilon_0>0$ small enough and $2<q_0<p_+(L)$, close enough to $p_+(L)$ so that for $r:=\frac{q_0 n}{2(1+\varepsilon_0)(n-q_0)}$, we have that $2<q_0<p_+(L)$, $q_0/2\leq r<\infty$, $w\in RH_{r'}$, and

$$
1+\frac{n}{2r}-\frac{n}{q_0}>0.
$$

If now $n\leq p_+(L)$, our condition over the weight w becomes $w\in A_\infty$. Then, we take $r>s_w$, and q_0 satisfying $\max\left\{2,\frac{2rp_+(L)}{p_+(L)+2r}\right\}<q_0<\min\{p_+(L),2r\}$ if $p_+(L)<\infty$ and $q_0=2r$ if $p_+(L)=\infty$. Therefore, we have that $2<q_0<p_+(L)$, $q_0/2\leq r<\infty$, and $w\in RH_{r'}$. Besides,

$$
1+\frac{n}{2r}-\frac{n}{q_0}>1-\frac{n}{p_+(L)}\geq 0.
$$

Hence, we have found the desired q_0 and r. Keeping these choices of q_0 and r we prove part (b). Using again (1.20) and Minkowski's integral inequality, we obtain

$$
\begin{aligned}
\widetilde{\mathcal{S}}f(x) \lesssim &\left(\int_0^\infty\left(\int_0^t\left(\int_{B(x,t)}|tsLe^{-s^2L}e^{-t^2L}f(y)|^2dy\right)^{\frac12}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac12}\\
&+\left(\int_0^\infty\left(\int_t^\infty\left(\int_{B(x,t)}|tsLe^{-s^2L}e^{-t^2L}f(y)|^2dy\right)^{\frac12}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac12} =: I+II.
\end{aligned}
$$

We first estimate I. Using that $s<t$ and applying the fact that $\{e^{-s^2L}\}_{s>0}\in\mathcal{F}_\infty(L^2\to L^2)$, we have

$$
I\leq\left(\int_0^\infty\left(\int_0^t\frac{s}{t}\left(\int_{B(x,t)}|e^{-s^2L}t^2Le^{-t^2L}f(y)|^2dy\right)^{\frac12}\frac{ds}{s}\right)^2\frac{dt}{t^{n+1}}\right)^{\frac12}
$$

$$\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_0^\infty \left(\int_0^t \frac{s}{t} \frac{ds}{s} \right)^2 \int_{B(x,2^{j+1}t)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}$$

$$\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_0^\infty \int_{B(x,2^{j+1}t)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Therefore, applying change of angles (Proposition 2.43), we get

$$\|I\|_{L^2(w)} \lesssim \sum_{j\geq 1} e^{-c4^j} \|\mathcal{S}_{\mathrm{H}} f\|_{L^2(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^2(w)}.$$

As for the second term, we first apply the fact that $\{e^{-t^2 L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^2)$, change the variable s into st, and apply Jensen's inequality. Next, we apply Minkowski's integral inequality and change the variable t into t/s. Hence, we have

$$\begin{aligned}
II &\lesssim \sum_{j\geq 1} e^{-c4^j} \left(\int_0^\infty \left(\int_1^\infty s^{-1} \left(\int_{B(x,2^{j+1}t)} |(st)^2 L e^{-(st)^2 L} f(y)|^{q_0} \frac{dy}{t^n} \right)^{\frac{1}{q_0}} \frac{ds}{s} \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} \int_1^\infty s^{-1+\frac{n}{q_0}} \left(\int_0^\infty \left(\int_{B(x,2^{j+1}t/s)} |t^2 L e^{-t^2 L} f(y)|^{q_0} \frac{dy}{t^n} \right)^{\frac{2}{q_0}} \frac{dt}{t} \right)^{\frac{1}{2}} \frac{ds}{s} \\
&=: \sum_{j\geq 1} e^{-c4^j} \int_1^\infty s^{-1+\frac{n}{q_0}} \mathcal{J}(x,s) \frac{ds}{s}.
\end{aligned}$$

Thus, applying first Minkowski's integral inequality, Proposition 2.61, and changing the variable t into $\sqrt{2}t$; next, applying the fact that $\{e^{-t^2 L}\}_{t>0} \in \mathcal{F}_\infty(L^2 \to L^{q_0})$, recalling our choices of q_0 and r and (4.80), and Proposition 2.43, we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \left(\int_1^\infty s^{-1+\frac{n}{q_0}} \mathcal{J}(x,s) \frac{ds}{s} \right)^2 w(x)dx \right)^{\frac{1}{2}} \\
&\qquad \lesssim \left(\int_1^\infty s^{-1-\frac{n}{2r}+\frac{n}{q_0}} \frac{ds}{s} \right) \left(\int_{\mathbb{R}^n} \int_0^\infty \left(\int_{B(x,2^{j+2}t)} |e^{-t^2 L} t^2 L e^{-t^2 L} f(y)|^{q_0} \frac{dy}{t^n} \right)^{\frac{2}{q_0}} \frac{dt}{t} w(x)dx \right)^{\frac{1}{2}} \\
&\qquad \lesssim \sum_{l\geq 1} e^{-c4^l} \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2^{j+l+3}t)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \right)^{\frac{1}{2}} \\
&\qquad \lesssim \left(\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2^{j+1}t)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} w(x)dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Using this and again applying Proposition 2.43, we get

$$\|II\|_{L^2(w)} \lesssim \sum_{j\geq 1} e^{-c4^j} \|\mathcal{S}_{\mathrm{H}} f\|_{L^2(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^2(w)}.$$

Gathering this and the estimate obtained for $\|I\|_{L^2(w)}$ gives us that, for all $w \in RH_{\left(\frac{p_+(L)^*}{2}\right)'}$,

$$\|\widetilde{\mathcal{S}} f\|_{L^2(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^2(w)},$$

which, from the observations made at the beginning, finishes the proof. □

Remark 4.81. *As we explain in Remark 3.22, we can extend Proposition 4.69 to all functions $f \in L^q(w)$ for $w \in A_\infty$ and $q \in \mathcal{W}_w(p_-(L), p_+(L))$.*

Now, we are ready to prove Proposition 4.63 and 4.65.

Proof of Proposition 4.63.

First of all note that if f is such that $\|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)} < \infty$, then, for $h := L^{-\frac{1}{2}} f$ we have that $h \in \dot{W}^{1,p}(w)$ (the space $\dot{W}^{1,p}(w)$ is defined as the completion of $\{h \in C_0^\infty(\mathbb{R}^n) : \nabla h \in L^p(w)\}$ under the semi-norm $\|h\|_{\dot{W}^{1,p}(w)} := \|\nabla h\|_{L^p(w)}$). Additionally, note that applying Proposition 4.69, Theorem 3.52, and [11, Theorem 6.2], for all $w \in A_\infty$ such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$ and $\widehat{p}_-(L) < p < \frac{p_+(L)}{s_w}$, we have that

$$\|\widetilde{\mathcal{S}}\sqrt{L}h\|_{L^p(w)} \approx \|\mathcal{S}_{\mathrm{H}}\sqrt{L}h\|_{L^p(w)} \lesssim \|\sqrt{L}h\|_{L^p(w)} \lesssim \|\nabla h\|_{L^p(w)}.$$

This gives us that

$$\widetilde{\mathcal{S}}\sqrt{L} : \dot{W}^{1,p}(w) \to L^p(w), \quad \forall\, \widehat{p}_-(L) < p < \frac{p_+(L)}{s_w}. \tag{4.82}$$

Therefore, if we show that, for every $\widehat{p}_-(L) < \widetilde{p} < \frac{q_+(L)}{s_w}$, $r_0 > r_w$, so that $r_w q_-(L) < r_0 q_-(L) < \frac{q_+(L)}{s_w}$, and for $p_0 := \max\left\{r_0, \frac{nr_0\widetilde{p}}{nr_0+\widetilde{p}}\right\}$,

$$\widetilde{\mathcal{S}}\sqrt{L} : \dot{W}^{1,p_0}(w) \to L^{p_0,\infty}(w), \tag{4.83}$$

then, by interpolation (see [18]), applying Proposition 4.69, and by the observation made at the beginning of the proof, we will conclude (4.64). Besides, note that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$ implies $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$ (recall that $\mathcal{W}_w(q_-(L), q_+(L)) \subset \mathcal{W}_w(p_-(L), p_+(L))$).

We fix $\widetilde{p}$ and r_0 satisfying the above restrictions. Additionally, we take r, $q_-(L) < r < 2$, close enough to $q_-(L)$ so that $rr_0 < \frac{q_+(L)}{s_w}$. Then, if we consider p_1 so that $\max\{rr_0, \widetilde{p}\} < p_1 < \frac{q_+(L)}{s_w}$, we have that $w \in A_{\frac{p_1}{r}} \cap RH_{\left(\frac{q_+(L)}{p_1}\right)'}$, and $p_1 > p_0$.

Recalling these choices of $\widetilde{p}$, r_0, r, p_1, and p_0, note that in order to prove (4.83) it suffices to show that, for every $\alpha > 0$ and $h \in \dot{W}^{1,p_0}(w)$,

$$w\left(\left\{x \in \mathbb{R}^n : \widetilde{\mathcal{S}}\sqrt{L}h(x) > \alpha\right\}\right) \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x)dx.$$

To this end, consider the following Calderón-Zygmund decomposition of h (see [11, Lemma 6.6]).

Lemma 4.84. *Let $n \geq 1$, $w \in A_\infty$, $\mu := wdx$, and $r_w < p_0 < \infty$ (with the possibility of taking $p_0 = 1$ if $r_w = 1$). Assume that $h \in \dot{W}^{1,p_0}(w)$, and let $\alpha > 0$. Then, one can find a collection of balls $\{B_i\}_{i\in\mathbb{N}}$ (with radii r_{B_i}), smooth functions b_i, and a function $g \in L^1_{loc}(w)$ such that*

$$h = g + \sum_{i\in\mathbb{N}} b_i$$

and the following properties hold

$$|\nabla g(x)| \leq C\alpha, \text{ for } \mu - a.e.\ x, \tag{4.85}$$

$$\operatorname{supp} b_i \subset B_i \quad \text{and} \quad \int_{B_i} |\nabla b_i(x)|^{p_0} w(x)dx \leq C\alpha^{p_0} w(B_i), \tag{4.86}$$

$$\sum_{i\in\mathbb{N}} w(B_i) \leq \frac{C}{\alpha^{p_0}} \int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x)dx, \tag{4.87}$$

$$\sum_{i\in\mathbb{N}} \mathbf{1}_{B_i} \le N, \tag{4.88}$$

where C and N depend only on the dimension, the doubling constant of μ, and p_0. In addition, for $1 \le q < (p_0)_w^*$, where $(p_0)_w^* = \frac{nr_w p_0}{nr_w - p_0}$ if $p_0 < nr_w$, and $(p_0)_w^* = \infty$ otherwise, we have

$$\left(\fint_{B_i} |b_i(x)|^q dw\right)^{\frac{1}{q}} \lesssim \alpha r_{B_i}. \tag{4.89}$$

Applying this lemma to our function h and our choice of p_0, and considering for $M \in \mathbb{N}$, arbitrarily large, and for every $i \in \mathbb{N}$, $B_{r_{B_i}} := (I - e^{-r_{B_i}^2 L})^M$ and $A_{r_{B_i}} := I - B_{r_{B_i}}$, we can write $b_i = B_{r_{B_i}} b_i + A_{r_{B_i}} b_i$. Hence,

$$h = g + \sum_{i\in\mathbb{N}} B_{r_{B_i}} b_i + \sum_{i\in\mathbb{N}} A_{r_{B_i}} b_i.$$

Then,

$$\begin{aligned} w\left(\left\{x \in \mathbb{R}^n : \widetilde{S}\sqrt{L}h(x) > \alpha\right\}\right) &\le w\left(\left\{x \in \mathbb{R}^n : \widetilde{S}\sqrt{L}g(x) > \frac{\alpha}{3}\right\}\right) \\ &+ w\left(\left\{x \in \mathbb{R}^n : \widetilde{S}\sqrt{L}\left(\sum_{i\in\mathbb{N}} A_{r_{B_i}} b_i\right)(x) > \frac{\alpha}{3}\right\}\right) + w\left(\left\{x \in \mathbb{R}^n : \widetilde{S}\sqrt{L}\left(\sum_{i\in\mathbb{N}} B_{r_{B_i}} b_i\right)(x) > \frac{\alpha}{3}\right\}\right) \\ &=: I + II + III. \end{aligned} \tag{4.90}$$

By our choice of p_1, we have that $p_1 \in \mathcal{W}_w(q_-(L), q_+(L)) \subset \mathcal{W}_w(p_-(L), p_+(L))$. Hence, applying Chebychev's inequality, (4.82), (4.85), (4.86), and (4.87), we obtain

$$\begin{aligned} I &\lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\widetilde{S}\sqrt{L}g(x)|^{p_1} w(x)dx \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\nabla g(x)|^{p_1} w(x)dx \\ &\lesssim \frac{1}{\alpha^{p_0}} \left(\int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x)dx + \alpha^{p_0} \sum_{i\in\mathbb{N}} w(B_i)\right) \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x)dx. \end{aligned} \tag{4.91}$$

In order to estimate the remaining terms, we take $1 < p < \infty$ and $u \in L^{p'}(w)$ such that $\|u\|_{L^{p'}(w)} = 1$. Besides, we denote by $\mathcal{M}^w$ the weighted maximal operator defined as in (2.88) but taking the supremum over balls instead of over cubes. Then, using a Kolmogorov type inequality and (4.87), we have that

$$\begin{aligned} \left(\sum_{i\in\mathbb{N}} \int_{B_i} \left(\mathcal{M}^w(|u|^{p'})(x)\right)^{\frac{1}{p'}} w(x)dx\right)^p &\lesssim \left(\int_{\cup_{i\in\mathbb{N}} B_i} \left(\mathcal{M}^w(|u|^{p'})(x)\right)^{\frac{1}{p'}} w(x)dx\right)^p \\ &\lesssim w(\cup_{i\in\mathbb{N}} B_i)\|u\|_{L^{p'}(w)}^p \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x)dx. \end{aligned} \tag{4.92}$$

Indeed, since $\mathcal{M}^w$ is of weak type $(1,1)$,

$$\begin{aligned} \int_{\cup_{i\in\mathbb{N}} B_i} \left(\mathcal{M}^w(|u|^{p'})(x)\right)^{\frac{1}{p'}} w(x)dx &= \frac{1}{p'} \int_0^\infty \lambda^{-1/p} w(\{x \in \cup_{i\in\mathbb{N}} B_i : \mathcal{M}^w(|u|^{p'})(x) > \lambda\})d\lambda \\ &\le \frac{1}{p'} w(\cup_{i\in\mathbb{N}} B_i) \int_0^a \lambda^{-1/p} d\lambda + \frac{1}{p'} \||u|^{p'}\|_{L^1(w)} \int_a^\infty \lambda^{-1/p-1} d\lambda \\ &= \frac{1}{p'} \left(p' a^{\frac{1}{p'}} w(\cup_{i\in\mathbb{N}} B_i) + p a^{-\frac{1}{p}} \||u|^{p'}\|_{L^1(w)}\right). \end{aligned}$$

Then, taking $a = \|u^{p'}\|_{L^1(w)} w(\cup_{i\in\mathbb{N}} B_i)^{-1}$, we conclude that

$$\int_{\cup_{i\in\mathbb{N}} B_i} \left(\mathcal{M}^w(|u|^{p'})(x)\right)^{\frac{1}{p'}} w(x)dx \leq p\|u\|_{L^{p'}(w)} w(\cup_{i\in\mathbb{N}} B_i)^{\frac{1}{p}}.$$

Moreover, note that $1 < \widetilde{p} < (p_0)_w^*$ and hence by (4.89),

$$\left(\fint_{B_i} |b_i(x)|^{\widetilde{p}} dw\right)^{\frac{1}{\widetilde{p}}} \lesssim \alpha r_{B_i}. \tag{4.93}$$

By our choice of $\widetilde{p}$, we have that $\widetilde{p} > 1$. In order to show that $1 \leq \widetilde{p} < (p_0)_w^*$, we first consider the case $n = 1$. Note that in this case $p_0 = r_0 > r_w$ which implies that $(p_0)_w^* = \infty$. Consequently $\widetilde{p} < (p_0)_w^*$. In the case $n \geq 2$, we observe that it suffices to consider the case $p_0 < nr_w$. In this case, if we suppose that $p_0 = r_0$ then $\widetilde{p} \leq \frac{nr_0}{n-1}$ and since $\frac{nr_0}{n-1} < \frac{nr_w r_0}{nr_w - r_0} = (p_0)_w^*$, we conclude the desired estimate. On the other hand, if $p_0 = \frac{nr_0\widetilde{p}}{nr_0+\widetilde{p}}$ then $(p_0)_w^* = \frac{nr_w p_0}{nr_w - p_0} = \frac{nr_w r_0 \widetilde{p}}{nr_w r_0 + (r_w - r_0)\widetilde{p}}$, (observe that $nr_w r_0 + (r_w - r_0)\widetilde{p} > 0$). Besides, $nr_w r_0 + (r_w - r_0)\widetilde{p} < nr_w r_0$, hence $\widetilde{p} < (p_0)_w^*$.

Therefore, in order to estimate II, we first apply Chebychev's inequality. Next, by (4.82), expanding the binomial, using that $\{\sqrt{t}\nabla_y e^{-tL}\}_{t>0} \in \mathcal{O}(L^{\widetilde{p}}(w) - L^{\widetilde{p}}(w))$ (see Proposition 1.41), by (1.11), (4.93), and (4.92) with $p = \widetilde{p}$, we have

$$\begin{aligned}
II &\lesssim \frac{1}{\alpha^{\widetilde{p}}} \int_{\mathbb{R}^n} \left| \widetilde{S}\sqrt{L}\left(\sum_{i\in\mathbb{N}} A_{r_{B_i}} b_i\right)(x)\right|^{\widetilde{p}} w(x)dx \\
&\lesssim \frac{1}{\alpha^{\widetilde{p}}} \int_{\mathbb{R}^n} \left| \nabla\left(\sum_{i\in\mathbb{N}} \sum_{k=1}^{M} C_{k,M} e^{-kr_{B_i}^2 L} b_i\right)(x)\right|^{\widetilde{p}} w(x)dx \\
&\lesssim \frac{1}{\alpha^{\widetilde{p}}} \sup_{\|u\|_{L^{\widetilde{p}'}(w)}=1} \left(\sum_{k=1}^{M} C_{k,M} \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} \left|\sqrt{k} r_{B_i} \nabla_y e^{-kr_{B_i}^2 L}\left(\frac{b_i}{r_{B_i}}\right)(x)\right| |u(x)| w(x)dx\right)^{\widetilde{p}} \\
&\lesssim \frac{1}{\alpha^{\widetilde{p}}} \sup_{\|u\|_{L^{\widetilde{p}'}(w)}=1} \left(\sum_{k=1}^{M} C_{k,M} \sum_{i\in\mathbb{N}} \sum_{j\geq 1} 2^{\frac{jnp_1}{r}} w(B_i) \left(\fint_{C_j(B_i)} \left|\sqrt{k} r_{B_i} \nabla_y e^{-kr_{B_i}^2 L}\left(\frac{b_i}{r_{B_i}}\right)(x)\right|^{\widetilde{p}} dw\right)^{\frac{1}{\widetilde{p}}} \left(\fint_{C_j(B_i)} |u(x)|^{\widetilde{p}'} dw\right)^{\frac{1}{\widetilde{p}'}}\right)^{\widetilde{p}} \\
&\lesssim \frac{1}{\alpha^{\widetilde{p}}} \sup_{\|u\|_{L^{\widetilde{p}'}(w)}=1} \left(\sum_{i\in\mathbb{N}} \sum_{j\geq 1} e^{-c4^j} w(B_i) \left(\fint_{B_i} \left|\frac{b_i(x)}{r_{B_i}}\right|^{\widetilde{p}} dw\right)^{\frac{1}{\widetilde{p}}} \inf_{x\in B_i} \left(\mathcal{M}^w(|u|^{\widetilde{p}'})(x)\right)^{\frac{1}{\widetilde{p}'}}\right)^{\widetilde{p}} \\
&\lesssim \sup_{\|u\|_{L^{\widetilde{p}'}(w)}=1} \left(\sum_{i\in\mathbb{N}} \int_{B_i} \left(\mathcal{M}^w(|u|^{\widetilde{p}'})(x)\right)^{\frac{1}{\widetilde{p}'}} w(x)dx\right)^{\widetilde{p}} \\
&\lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x)dx.
\end{aligned} \tag{4.94}$$

Next, we estimate III. Note that,

$$III \lesssim w\left(\bigcup_{i\in\mathbb{N}} 16B_i\right) + w\left(\left\{x \in \mathbb{R}^n \setminus \bigcup_{i\in\mathbb{N}} 16B_i : \widetilde{S}\sqrt{L}\left(\sum_{i\in\mathbb{N}} B_{r_{B_i}} b_i\right)(x) > \frac{\alpha}{3}\right\}\right) = III_1 + III_2. \tag{4.95}$$

Applying (4.87) we have that

$$III_1 \lesssim \frac{1}{\alpha^{p_0}} \int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x)dx. \tag{4.96}$$

Hence it just remains to control III_2. Applying Chebychev's inequality, we obtain

$$III_2 \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n \setminus \bigcup_{i\in\mathbb{N}} 16B_i} \left(\int_0^\infty \int_{B(x,t)} \left| tLe^{-t^2L} \left(\sum_{i\in\mathbb{N}} B_{r_{B_i}} b_i \right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p_1}{2}} w(x)dx$$

$$\lesssim \frac{1}{\alpha^{p_1}} \sup_{\|u\|_{L^{p_1'}(w)}=1} \left(\sum_{i\in\mathbb{N}} \sum_{j\geq 4} \left(\int_{C_j(B_i)} \left(\int_0^\infty \int_{B(x,t)} \left| tLe^{-t^2L} \left(B_{r_{B_i}} b_i \right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p_1}{2}} w(x)dx \right)^{\frac{1}{p_1}} \|u\mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(w)} \right)^{p_1}$$

$$=: \frac{1}{\alpha^{p_1}} \sup_{\|u\|_{L^{p_1'}(w)}=1} \left(\sum_{i\in\mathbb{N}} \sum_{j\geq 4} \mathcal{III}_{ij} \|u\mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(w)} \right)^{p_1}. \tag{4.97}$$

Splitting the integral in t (recall that $j \geq 4$), we have

$$\mathcal{III}_{ij} \lesssim \left(\int_{C_j(B_i)} \left(\int_0^{2^{j-2}r_{B_i}} \int_{B(x,t)} \left| tLe^{-t^2L} \left(B_{r_{B_i}} b_i \right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p_1}{2}} w(x)dx \right)^{\frac{1}{p_1}}$$

$$+ \left(\int_{C_j(B_i)} \left(\int_{2^{j-2}r_{B_i}}^\infty \int_{B(x,t)} \left| t^2Le^{-t^2L} \left(B_{r_{B_i}} \left(\frac{b_i}{r_{B_i}} \right) \right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{p_1}{2}} w(x)dx \right)^{\frac{1}{p_1}} = \mathcal{III}_{ij}^1 + \mathcal{III}_{ij}^2.$$

We first estimate $\mathcal{III}_{ij}^1$. Recall that $w \in A_{\frac{p_1}{r}} \cap RH_{\left(\frac{q_+(L)}{p_1}\right)'}$. Hence, taking q_0, $\max\{2, p_1\} < q_0 < q_+(L)$, close enough to $q_+(L)$ so that $w \in RH_{\left(\frac{q_0}{p_1}\right)'}$, applying Jensen's inequality, Fubini's theorem, and noticing that for $x \in C_j(B_i)$ and $0 < t \leq 2^{j-2}r_{B_i}$ we have that $B(x,t) \subset 2^{j+2}B_i \setminus 2^{j-1}B_i$, we get

$$\mathcal{III}_{ij}^1 \lesssim |2^{j+1}B_i|^{-\frac{1}{q_0}} w(2^{j+1}B_i)^{\frac{1}{p_1}} \left(\int_{C_j(B_i)} \left(\int_0^{2^{j-2}r_{B_i}} \int_{B(x,t)} \left| tLe^{-t^2L} \left(B_{r_{B_i}} b_i \right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{q_0}{2}} dx \right)^{\frac{1}{q_0}}$$

$$\lesssim |2^{j+1}B_i|^{-\frac{1}{q_0}} w(2^{j+1}B_i)^{\frac{1}{p_1}} \left(\int_{C_j(B_i)} \int_0^{2^{j-2}r_{B_i}} \left(\frac{2^j r_{B_i}}{t} \right)^{\frac{q_0}{2}-1} \int_{B(x,t)} \left| tLe^{-t^2L} \left(B_{r_{B_i}} b_i \right)(y) \right|^{q_0} \frac{dy\,dt}{t^{n+1}} dx \right)^{\frac{1}{q_0}}$$

$$\lesssim |2^{j+1}B_i|^{-\frac{1}{q_0}} w(2^{j+1}B_i)^{\frac{1}{p_1}} \left(\int_0^{2^{j-2}r_{B_i}} \left(\frac{2^j r_{B_i}}{t} \right)^{\frac{q_0}{2}-1} t^{-q_0} \int_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| t^2Le^{-t^2L} \left(B_{r_{B_i}} b_i \right)(y) \right|^{q_0} \frac{dy\,dt}{t} \right)^{\frac{1}{q_0}}.$$

We estimate the integral in y by using functional calculus. We use the notation in [3] and [11, Section 7]. We write $\vartheta \in [0, \pi/2)$ for the supremum of $|\arg(\langle Lf, f\rangle_{L^2(\mathbb{R}^n)})|$ over all f in the domain of L. Let $0 < \vartheta < \theta < \nu < \mu < \pi/2$ and note that, for a fixed $t > 0$, $\phi(z,t) := e^{-t^2z}(1 - e^{-r_{B_i}^2 z})^M$ is holomorphic in the open sector $\Sigma_\mu = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \mu\}$ and satisfies $|\phi(z,t)| \lesssim |z|^M (1 + |z|)^{-2M}$ (with implicit constant depending on μ, $t > 0$, r_{B_i}, and M) for every $z \in \Sigma_\mu$. In order to see this, note that, in general for $k > 0$ and $-\pi/2 < \alpha < \pi/2$, we have

$$|1 - e^{-ke^{\pm i\alpha}}| = |1 - e^{-k(\cos(\alpha) \pm i\sin(\alpha))}| = |1 - e^{-k\cos(\alpha)}\cos(k\sin(\alpha)) \mp ie^{-k\cos(\alpha)}\sin(k\sin(\alpha))|$$

$$= \left(\left(1 - e^{-k\cos(\alpha)}\cos(k\sin(\alpha))\right)^2 + \left(e^{-k\cos(\alpha)}\sin(k\sin(\alpha))\right)^2 \right)^{\frac{1}{2}}.$$

Now, consider $f(x) := e^{-x}\cos\left(x\frac{\sin(\alpha)}{\cos(\alpha)}\right)$ and $g(x) := e^{-x}\sin\left(x\frac{\sin(\alpha)}{\cos(\alpha)}\right)$, we have that $f(0) = 1$, $f(k\cos(\alpha)) = e^{-k\cos(\alpha)}\cos(k\sin(\alpha))$, $g(0) = 0$, and $g(k\cos(\alpha)) = e^{-k\cos(\alpha)}\sin(k\sin(\alpha))$. Besides, for all $x > 0$

$$|f'(x)| \leq 1 + |\tan(\alpha)| \quad \text{and} \quad |g'(x)| \leq 1 + |\tan(\alpha)|.$$

Then, applying the mean value theorem, we conclude that, for all $k > 0$ and $-\pi/2 < \alpha < \pi/2$

$$|1 - e^{-ke^{\pm i\alpha}}| \leq 4k. \tag{4.98}$$

Using this for $z \in \Sigma_\mu$ and $t > 0$, we obtain

$$|\phi(z,t)| = \left|e^{-t^2|z|e^{i\arg(z)}}\right| \left|1 - e^{-r_{B_i}^2|z|e^{i\arg z}}\right|^M \leq 4^M|z|^M r_{B_i}^{2M} e^{-t^2\cos(\arg z)(1+|z|)} e^{t^2\cos(\arg z)}$$
$$\lesssim 4^M r_{B_i}^{2M} \frac{e^{t^2}}{t^{4M}\cos(\mu)^{2M}} \frac{|z|^M}{(1+|z|)^{2M}},$$

with the implicit constant depending on M. Hence, we can write

$$\phi(L,t) = \int_\Gamma e^{-zL}\eta(z,t)dz, \qquad \text{where} \quad \eta(z,t) = \int_\gamma e^{\zeta z}\phi(\zeta,t)d\zeta.$$

Here $\Gamma = \partial\Sigma_{\frac{\pi}{2}-\theta}$ with positive orientation (although orientation is irrelevant for our computations) and $\gamma = \mathbb{R}_+ e^{i\,\mathrm{sign}(\mathrm{Im}(z))\,\nu}$. It is not difficult to see that for every $z \in \Gamma$,

$$|\eta(z,t)| \lesssim \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}}.$$

Indeed, note that $\pi/2 < \nu + \pi/2 - \theta < \pi$. Consequently, $\cos(\nu + \pi/2 - \theta) < 0$. Then, by (4.98), for every $z \in \Gamma$, $\zeta \in \gamma$, and for $c_0 := \min\{\cos(\nu), |\cos(\nu + \pi/2 - \theta)|\} > 0$, we have

$$|e^{\zeta z}\phi(\zeta,t)| \lesssim e^{|\zeta||z|\cos(\nu+\pi/2-\theta)} e^{-t^2|\zeta|\cos(\nu)}|\zeta|^M r_{B_i}^{2M} \leq |\zeta|^M r_{B_i}^{2M} e^{-c_0|\zeta|(t^2+|z|)},$$

with the implicit constant depending on M. Therefore, for every $z \in \Gamma$

$$|\eta(z,t)| \lesssim r_{B_i}^{2M}\int_0^\infty l^M e^{-c_0(t^2+|z|)l}\,dl \leq r_{B_i}^{2M}\int_0^{\frac{1}{|z|+t^2}} l^M e^{-c_0(t^2+|z|)l}\,dl + r_{B_i}^{2M}\int_{\frac{1}{|z|+t^2}}^\infty l^M e^{-c_0(t^2+|z|)l}\,dl$$
$$\lesssim \frac{r_{B_i}^{2M}}{|z|+t^2}\int_0^{\frac{1}{|z|+t^2}} l^{M-1}\,dl + \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+2}}\int_{\frac{1}{|z|+t^2}}^\infty \frac{dl}{l^2} \approx \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}},$$

with the implicit constant depending on M, ν, and θ.

Thus, we can write

$$\left(\int_{2^{j+2}B_i\setminus 2^{j-1}B_i} \left|t^2Le^{-t^2L}B_{r_{B_i}}(b_i)(y)\right|^{q_0} dy\right)^{\frac{1}{q_0}}$$
$$\lesssim \int_\Gamma \left(\int_{2^{j+2}B_i\setminus 2^{j-1}B_i} \left|\frac{z}{2}Le^{-\frac{z}{2}L}\left(e^{-\frac{z}{2}L}b_i\right)(y)\right|^{q_0} dy\right)^{\frac{1}{q_0}} \frac{t^2}{|z|}\frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}}|dz|$$
$$\lesssim \sum_{l=1}^{j-3}\int_\Gamma \left(\int_{2^{j+2}B_i\setminus 2^{j-1}B_i} \left|\frac{z}{2}Le^{-\frac{z}{2}L}\left(\mathbf{1}_{C_l(B_i)}e^{-\frac{z}{2}L}b_i\right)(y)\right|^{q_0} dy\right)^{\frac{1}{q_0}} \frac{t^2}{|z|}\frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}}|dz|$$
$$+ \sum_{l\geq j-2}\int_\Gamma \left(\int_{2^{j+2}B_i\setminus 2^{j-1}B_i} \left|\frac{z}{2}Le^{-\frac{z}{2}L}\left(\mathbf{1}_{C_l(B_i)}e^{-\frac{z}{2}L}b_i\right)(y)\right|^{q_0} dy\right)^{\frac{1}{q_0}} \frac{t^2}{|z|}\frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}}|dz|.$$

Note now that since $j \geq 4$, for $1 \leq l \leq j-3$ we have that $d(2^{j+2}B_i \setminus 2^{j-1}B_i, C_l(B_i)) \geq 2^{j-2}r_{B_i} \geq 2^{l+1}r_{B_i}$. Then, in that case, applying the fact that $\frac{z}{2}Le^{-\frac{z}{2}L}$ satisfies $L^r(\mathbb{R}^n) - L^{q_0}(\mathbb{R}^n)$ off-diagonal estimates (see [3]), splitting the exponential term, using that $w \in A_{\frac{p_1}{r}}$, changing the variable s into $4^j r_{B_i}^2/s^2$, and applying that $e^{-\frac{z}{2}L} \in \mathcal{O}(L^{\widetilde{p}}(w) - L^{p_1}(w))$ (see [10, 11]), and by (4.93), we obtain

$$
\begin{aligned}
&\int_\Gamma \left(\int_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| \frac{z}{2} L e^{-\frac{z}{2}L} \left(\mathbf{1}_{C_l(B_i)} e^{-\frac{z}{2}L} b_i \right)(y) \right|^{q_0} dy \right)^{\frac{1}{q_0}} \frac{t^2}{|z|} \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}} |dz| \\
&\lesssim \int_\Gamma \left(\int_{C_l(B_i)} \left| e^{-\frac{z}{2}L} b_i(y) \right|^r dy \right)^{\frac{1}{r}} |z|^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-c\frac{4^j r_{B_i}^2}{|z|}} \frac{t^2}{|z|} \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}} |dz| \\
&\lesssim (2^l r_{B_i})^{\frac{n}{r}} \int_\Gamma \left(\fint_{C_l(B_i)} \left| e^{-\frac{z}{2}L} b_i(y) \right|^{p_1} dw \right)^{\frac{1}{p_1}} |z|^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-c\frac{4^j r_{B_i}^2}{|z|}} e^{-c\frac{4^l r_{B_i}^2}{|z|}} \frac{t^2}{|z|} \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}} |dz| \\
&\lesssim 2^{l\theta_1} (2^l r_{B_i})^{\frac{n}{r}} \left(\fint_{B_i} |b_i(y)|^{\widetilde{p}} dw \right)^{\frac{1}{\widetilde{p}}} \int_0^\infty \Upsilon\left(\frac{2^l r_{B_i}}{s^{\frac{1}{2}}} \right)^{\theta_2} s^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-c\frac{4^j r_{B_i}^2}{s}} e^{-c\frac{4^l r_{B_i}^2}{s}} t^2 \frac{r_{B_i}^{2M}}{(s+t^2)^{M+1}} \frac{ds}{s} \\
&\lesssim \alpha r_{B_i} 2^{l\left(\theta_1+\frac{n}{r}\right)} r_{B_i}^{\frac{n}{q_0}} 2^{-jn\left(\frac{1}{r}-\frac{1}{q_0}\right)} \int_0^\infty \Upsilon\left(\frac{2^l s}{2^j} \right)^{\theta_2} s^{n\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-cs^2} e^{-c\frac{4^l s^2}{4^j}} t^2 \frac{r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+t^2)^{M+1}} \frac{ds}{s},
\end{aligned}
$$

recall that $\Upsilon(u) = \max\{u, u^{-1}\}$.

If we now consider $l \geq j-2$, in this case, we do not have distance between $2^{j+2}B_i \setminus 2^{j-1}B_i$ and $C_l(B_i)$, but we do have between $C_l(B_i)$ and B_i. Indeed, since $l \geq j-2 \geq 2$, we have that $d(C_l(B_i), B_i) > 2^{l-1} r_{B_i} \geq 2^{j-3} r_{B_i}$. Hence, proceeding as in the above computation, we obtain

$$
\begin{aligned}
&\int_\Gamma \left(\int_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| \frac{z}{2} L e^{-\frac{z}{2}L} \left(\mathbf{1}_{C_l(B_i)} e^{-\frac{z}{2}L} b_i \right)(y) \right|^{q_0} dy \right)^{\frac{1}{q_0}} \frac{t^2}{|z|} \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}} |dz| \\
&\lesssim \int_\Gamma \left(\int_{C_l(B_i)} \left| e^{-\frac{z}{2}L} b_i(y) \right|^r dy \right)^{\frac{1}{r}} |z|^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q_0}\right)} \frac{t^2}{|z|} \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}} |dz| \\
&\lesssim (2^l r_{B_i})^{\frac{n}{r}} \int_\Gamma \left(\fint_{C_l(B_i)} \left| e^{-\frac{z}{2}L} b_i(y) \right|^{p_1} dw \right)^{\frac{1}{p_1}} |z|^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q_0}\right)} \frac{t^2}{|z|} \frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}} |dz| \\
&\lesssim 2^{l\theta_1} (2^l r_{B_i})^{\frac{n}{r}} \left(\fint_{B_i} |b_i(y)|^{\widetilde{p}} dw \right)^{\frac{1}{\widetilde{p}}} \int_0^\infty \Upsilon\left(\frac{2^l r_{B_i}}{s^{\frac{1}{2}}} \right)^{\theta_2} s^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-c\frac{4^l r_{B_i}^2}{s}} t^2 \frac{r_{B_i}^{2M}}{(s+t^2)^{M+1}} \frac{ds}{s} \\
&\lesssim \alpha r_{B_i} 2^{l\theta_1} (2^l r_{B_i})^{\frac{n}{r}} \int_0^\infty \Upsilon\left(\frac{2^l r_{B_i}}{s^{\frac{1}{2}}} \right)^{\theta_2} s^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-c\frac{4^j r_{B_i}^2}{s}} e^{-c\frac{4^l r_{B_i}^2}{s}} t^2 \frac{r_{B_i}^{2M}}{(s+t^2)^{M+1}} \frac{ds}{s} \\
&\lesssim \alpha r_{B_i} 2^{l\left(\theta_1+\frac{n}{r}\right)} r_{B_i}^{\frac{n}{q_0}} 2^{-jn\left(\frac{1}{r}-\frac{1}{q_0}\right)} \int_0^\infty \Upsilon\left(\frac{2^l s}{2^j} \right)^{\theta_2} s^{n\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-cs^2} e^{-c\frac{4^l s^2}{4^j}} t^2 \frac{r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+t^2)^{M+1}} \frac{ds}{s}.
\end{aligned}
$$

Next, changing the variable t into $2^j r_{B_i} t$, we have for $\widetilde{M} > 0$ large enough to be chosen later,

$$
\begin{aligned}
&\left(\int_0^{2^{j-2} r_{B_i}} \left(\frac{2^j r_{B_i}}{t} \right)^{\frac{q_0}{2}-1} t^{-q_0} \left(\int_0^\infty \Upsilon\left(\frac{2^l s}{2^j} \right)^{\theta_2} s^{n\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-cs^2} e^{-c\frac{4^l s^2}{4^j}} t^2 \frac{r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+t^2)^{M+1}} \frac{ds}{s} \right)^{q_0} \frac{dt}{t} \right)^{\frac{1}{q_0}} \\
&\lesssim 2^{-j(2M+1)} r_{B_i}^{-1} \left(\int_0^1 t^{1+\frac{q_0}{2}} \left(\int_0^\infty \Upsilon\left(\frac{2^l s}{2^j} \right)^{\theta_2} s^{n\left(\frac{1}{r}-\frac{1}{q_0}\right)} e^{-cs^2} e^{-c\frac{4^l s^2}{4^j}} \frac{1}{(1/s^2+t^2)^{M+1}} \frac{ds}{s} \right)^{q_0} \frac{dt}{t} \right)^{\frac{1}{q_0}} \\
&\lesssim 2^{-l\left(2\widetilde{M}-\theta_2\right)} 2^{-j\left(2M+1-\theta_2-2\widetilde{M}\right)} r_{B_i}^{-1} \left(\left(\int_0^1 t^{1+\frac{q_0}{2}} \left(\int_0^1 s^{n\left(\frac{1}{r}-\frac{1}{q_0}\right)-2\widetilde{M}+2M+2-\theta_2} \frac{ds}{s} \right)^{q_0} \frac{dt}{t} \right)^{\frac{1}{q_0}} \right. \\
&\qquad \left. + \left(\int_0^1 t^{1+\frac{q_0}{2}} \left(\int_1^\infty s^{n\left(\frac{1}{r}-\frac{1}{q_0}\right)-2\widetilde{M}+2M+2+\theta_2} e^{-cs^2} \frac{ds}{s} \right)^{q_0} \frac{dt}{t} \right)^{\frac{1}{q_0}} \right).
\end{aligned}
$$

Therefore, taking $2\widetilde{M} = \theta_1 + \theta_2 + \frac{n}{r} + 1$, and $2M > \theta_1 + 2\theta_2 + \frac{n}{r}$, we have

$$\mathcal{III}_{ij}^1 \lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_1}} 2^{-j\left(2M+\frac{n}{r}+1-\theta_2-2\widetilde{M}\right)} \sum_{l\geq 1} 2^{-l} \lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_1}} 2^{-j(2M-\theta_1-2\theta_2)}. \tag{4.99}$$

In order to estimate $\mathcal{III}_{ij}^2$, we consider $\theta_M := \sqrt{M+2}$ and $B_{r_{B_i},t} := (e^{-t^2L} - e^{-(t^2+r_{B_i}^2)L})^M$. Hence, applying the fact that $\{t^2Le^{-t^2L}\}_{t>0} \in \mathcal{F}(L^r - L^2)$, Proposition 1.37 with $s = r_{B_i}$ and $p = r$, Lemma 1.35, and next using that $w \in A_{\frac{p_1}{r}}$, applying that $\{e^{-tL}\}_{t>0} \in \mathcal{O}(L^{\widetilde{p}}(w) - L^{p_1}(w))$, and by (4.93), we obtain

$$\begin{aligned}
&\left(\int_{B(x,\theta_M t)} \left| t^2Le^{-t^2L}B_{r_{B_i},t}\left(e^{-t^2L}\left(\mathbf{1}_{B(x,9\theta_M t)}\frac{b_i}{r_{B_i}}\right)\right)(y)\right|^2 \frac{dy}{t^n}\right)^{\frac{1}{2}}\\
&\quad\lesssim \left(\frac{r_{B_i}^2}{t^2}\right)^M \sum_{l\geq 1}\left(\int_{C_l(B(x,9\theta_M t))}\left|e^{-t^2L}\left(\mathbf{1}_{B(x,9\theta_M t)}\frac{b_i}{r_{B_i}}\right)(y)\right|^r \frac{dy}{t^n}\right)^{\frac{1}{r}}\\
&\quad\lesssim \left(\frac{r_{B_i}^2}{t^2}\right)^M \sum_{l\geq 1} 2^{\frac{ln}{r}}\left(\fint_{C_l(B(x,9\theta_M t))}\left|e^{-t^2L}\left(\mathbf{1}_{B(x,9\theta_M t)}\frac{b_i}{r_{B_i}}\right)(y)\right|^{p_1} dw\right)^{\frac{1}{p_1}}\\
&\quad\lesssim \left(\frac{r_{B_i}^2}{t^2}\right)^M \sum_{l\geq 1} e^{-c4^l}\left(\fint_{B(x,9\theta_M t)}\left|\frac{b_i(y)}{r_{B_i}}\right|^{\widetilde{p}} dw\right)^{\frac{1}{\widetilde{p}}}\\
&\quad\lesssim \alpha\left(\frac{r_{B_i}^2}{t^2}\right)^M \left(\frac{w(B_i)}{w(B(x,9\theta_M t))}\right)^{\frac{1}{\widetilde{p}}}.
\end{aligned}$$

Therefore, changing the variable t into $t\theta_M$ and noticing that, for $x \in C_j(B_i)$ and $t > \frac{2^{j-2}r_{B_i}}{\theta_M}$, we have that $B_i \subset B(x, 9\theta_M t)$, using the estimate above, we get

$$\begin{aligned}
\mathcal{III}_{ij}^2 &\lesssim \left(\int_{C_j(B_i)}\left(\int_{\frac{2^{j-2}r_{B_i}}{\theta_M}}^\infty \int_{B(x,\theta_M t)}\left|t^2Le^{-t^2L}B_{r_{B_i},t}\left(e^{-t^2L}\left(\frac{b_i}{r_{B_i}}\right)\right)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{p_1}{2}} w(x)dx\right)^{\frac{1}{p_1}}\\
&\lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_1}}\left(\int_{\frac{2^{j-2}r_{B_i}}{\theta_M}}^\infty\left(\frac{r_{B_i}^2}{t^2}\right)^{2M}\frac{dt}{t}\right)^{\frac{1}{2}} \lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_1}}2^{-j2M}.
\end{aligned}$$

This and (4.99) imply that $\mathcal{III}_{ij} \lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_1}} 2^{-j(2M-\theta_1-2\theta_2)}$. Hence, in view of (4.97), and by (1.11) and (4.92) with $p = p_1$, taking $2M > \frac{np_1}{r} + \theta_1 + 2\theta_2$, we obtain that

$$\begin{aligned}
III_2 &\lesssim \sup_{\|u\|_{L^{p_1'}(w)}=1}\left(\sum_{i\in\mathbb{N}} w(B_i)\inf_{x\in B_i}\left(\mathcal{M}^w(|u|^{p_1'})(x)\right)^{\frac{1}{p_1'}}\sum_{j\geq 4}2^{-j\left(2M-\frac{np_1}{r}-\theta_1-2\theta_2\right)}\right)^{p_1}\\
&\lesssim \sup_{\|u\|_{L^{p_1'}(w)}=1}\left(\sum_{i\in\mathbb{N}}\int_{B_i}\left(\mathcal{M}^w(|u|^{p_1'})(x)\right)^{\frac{1}{p_1'}} w(x)dx\right)^{p_1} \lesssim \frac{1}{\alpha^{p_0}}\int_{\mathbb{R}^n}|\nabla h(x)|^{p_0}w(x)dx.
\end{aligned}$$

Plugging this and (4.96) into (4.95) gives us $III \lesssim \alpha^{-p_0}\int_{\mathbb{R}^n}|\nabla h(x)|^{p_0}w(x)dx$. Hence, by this, (4.94), (4.91), and (4.90), we conclude (4.83).

To complete the proof note that for $\max\left\{r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)}\right\} < p < \frac{p_+(L)}{s_w}$ and $q \in \mathcal{W}_w(q_-(L), q_+(L))$, if we take $f \in \mathbb{H}^p_{\nabla L^{-1/2},q}(w)$, by (4.64), we have that

$$\|\mathcal{S}_{\mathrm{H}}f\|_{L^p(w)} \lesssim \|\nabla L^{-\frac{1}{2}}f\|_{L^p(w)},$$

consequently $f \in \mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$. $\square$

Proof of Proposition 4.65.

For $w \in A_\infty$ such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$, and $q \in \mathcal{W}_w(q_-(L), q_+(L)) \subset \mathcal{W}_w(p_-(L), p_+(L))$, if we take $p \in \mathcal{W}_w(q_-(L), q_+(L))$ and $f \in \mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$, applying Theorems 1.34 and 4.59, we obtain

$$\|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \approx \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)} = \|f\|_{\mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)}. \tag{4.100}$$

On the other hand, for $0 < p \leq 1$ by Propositions 4.68, part (b), and 4.40, part (a), we have

$$\|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)} \lesssim \|f\|_{\mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)}.$$

Therefore, applying Theorem 4.13 with $p_0 = 1$ and $p_1 \in \mathcal{W}_w(q_-(L), q_+(L))$, we conclude that, for all $0 < p < \frac{q_+(L)}{s_w}$, $q \in \mathcal{W}_w(q_-(L), q_+(L))$, and $f \in \mathbb{H}^p_{\mathcal{S}_{\mathrm{H}},q}(w)$,

$$\|\nabla L^{-\frac{1}{2}} f\|_{L^p(w)} \lesssim \|\mathcal{S}_{\mathrm{H}} f\|_{L^p(w)},$$

consequently $f \in \mathbb{H}^p_{\nabla L^{-1/2},q}(w)$. $\square$

Appendix A
AMALGAM SPACES

Amalgam spaces were first defined by Norbert Wiener in 1926, in the formulation of his generalized harmonic analysis. Although, he considered the particular cases $W(L^1, \ell^2)$ and $W(L^2, L^\infty)$ in [80], and $W(L^1, L^\infty)$ and $W(L^\infty, L^1)$ in [81], in general, for $1 \leq p, q < \infty$, the amalgam space $W(L^p, L^q)$ is defined as

$$W(L^p, L^q) := \left\{ f \in L^p_{loc}(\mathbb{R}) : \left(\sum_{n \in \mathbb{Z}} \left(\int_n^{n+1} |f(t)|^p dt \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}.$$

A significant difference in considering amalgam spaces instead of L^p spaces is that amalgam spaces give information about the local, L^q, and global, L^p, properties of the functions, while L^p spaces do not make that distinction.

A generalization of the definition of amalgam spaces for Banach function spaces was done by Feichtinger (see for instance [43] and [42]). For B and C Banach function spaces on a locally compact group G, satisfying certain conditions, he defined spaces $W(B, C)$ of distributions. The important thing is that we have equivalence of continuous and discrete norms on those spaces. This has been an important tool in applications. We refer to [52] for a deeper discussion on amalgam spaces in the real line.

Going on in the historical background of amalgam spaces, we highlight the paper of Holland, [57], that appears to be the first methodical study on amalgam spaces. After that there were important studies on amalgam spaces, for example, by Bertrandias, Datry, and Dupuis [21], Stewart [77], and Busby and Smith [25]. For a complete survey on amalgam spaces see [46].

A natural definition of amalgam spaces in dimension $n \geq 2$ is

$$(L^p, L^q)(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \left(\int_{\mathbb{R}^n} \|\mathbf{1}_{B(x,1)} f\|^q_{L^p} \, dx \right)^{\frac{1}{q}} < \infty \right\}.$$

Besides, for $1 \leq \alpha \leq \infty$, the subspace $(L^p, L^q)^\alpha(\mathbb{R}^n)$ of $(L^p, L^q)(\mathbb{R}^n)$ is defined in [45] by

$$(L^p, L^q)^\alpha(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{(L^p, L^q)^\alpha(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{(L^p, L^q)^\alpha(\mathbb{R}^n)} := \sup_{r>0} \left(\int_{\mathbb{R}^n} \left(|B(x,r)|^{\frac{1}{\alpha} - \frac{1}{p} - \frac{1}{q}} \|\mathbf{1}_{B(x,r)} f\|_{L^p} \right)^q dx \right)^{\frac{1}{q}}.$$

In [14] retracts of tent spaces called slice-spaces, are used. It turns out that they are closely related with amalgam spaces. Let us generalize their definition. For each $t > 0$ and $0 < p, r < \infty$, the slice-space $(E^p_r)_t$ is defined as the following set:

$$(E^p_r)_t := \left\{ f \in L^r_{loc}(\mathbb{R}^n) : \left(\fint_{B(x,t)} |f(y)|^r dy \right)^{\frac{1}{r}} \in L^p(\mathbb{R}^n) \right\}$$

with

$$\|f\|_{(E^p_r)_t} = \left\| \left(\fint_{B(\cdot,t)} |f(y)|^r dy \right)^{\frac{1}{r}} \right\|_{L^p(\mathbb{R}^n)}.$$

Besides, consider the weak slice-spaces

$$(wE^p_r)_t := \left\{ f \in L^r_{loc}(\mathbb{R}^n) : \left(\fint_{B(x,t)} |f(y)|^r dy \right)^{\frac{1}{r}} \in L^{p,\infty}(\mathbb{R}^n) \right\}.$$

with

$$\|f\|_{(wE^p_r)_t} = \left\| \left(\fint_{B(\cdot,t)} |f(y)|^r dy \right)^{\frac{1}{r}} \right\|_{L^{p,\infty}(\mathbb{R}^n)}.$$

For $1 \le r, p < \infty$, note that, for $n = 1$, $(E^p_r)_1 = W(L^r, L^p)$, and for $n \ge 2$, $(E^p_r)_1 = (L^r, L^p)(\mathbb{R}^n)$. Furthermore, for $p \in [r, \infty)$, since $\|f\|_{(E^p_r)_t} \le \|f\|_{L^p}$, for all $t > 0$, $(E^p_r)_t = (L^r, L^p)^p(\mathbb{R}^n)$.

Boundedness on amalgam spaces of the Hardy Littlewood maximal operator, of Calderón-Zygmund operators, of maximal fractional operators, Riesz potentials, etc, has been studied. See for instance [28], [34], [44], [63]. From Lemmas 2.29, 2.21 and 2.36, and Corollary 2.40, we can obtain easily boundedness of those operators on slice-spaces, and, therefore, on amalgam spaces. This significantly simplifies the previous proofs on this issue.

Let $1 \le r < \infty$. Let $t > 0$. Consider the applications i_t and π_t in [14]: for $f : \mathbb{R}^n \to \mathbb{C}$,

$$i_t(f)(x, s) = f(x)\mathbf{1}_{[t,et]}(s),$$

and for $G : \mathbb{R}^{n+1}_+ \to \mathbb{C}$,

$$\pi_t(G)(x) = \int_t^{et} G(x, s)\frac{ds}{s}.$$

It is easy to see that

$$\pi_t \circ i_t(f) = f.$$

Lemma A.1. *Let $0 < p < \infty$ and $1 \le r < \infty$. Then $i_t : (E^p_r)_t \to T^p_r$ and $\pi_t : T^p_r \to (E^p_r)_t$ are bounded with the norms being uniform with respect to t. In particular, the slice-spaces $(E^p_r)_t$ are retracts of T^p_r. The same happens for the weak slice-spaces, they are retracts of the weak tent spaces.*

Proof. For the slice-spaces when $r = 2$, this is observed without proof in [14]. The proof is the same for all (weak) slice-spaces. It suffices to note that

$$\left(\fint_{B(x,t)} |\pi_t(G)(y)|^r dy \right)^{\frac{1}{r}} \le C\mathcal{A}_r(G)(x)$$

and

$$\mathcal{A}_r(i_t(f))(x) \le C \left(\fint_{B(x,et)} |f(y)|^r dy \right)^{\frac{1}{r}}$$

for some dimensional constants C, and to use the norm comparison below for the slice-spaces and similarly for the weak slice-spaces. □

Lemma A.2. *If $0 < t, s < \infty$ with $t \sim s$, $1 \le r < \infty$ and $p \in (0, \infty)$, then $(E^p_r)_t = (E^p_r)_s$ with*

$$\|f\|_{(E^p_r)_t} \sim_{n,p} \|f\|_{(E^p_r)_s}$$

For any linear operator T on functions on $\mathbb{R}^n$, if $\mathcal{T}$ is its extension to functions on $\mathbb{R}^{n+1}$ by tensorisation, then we have $T = \pi_t \circ \mathcal{T} \circ i_t$. Hence the boundedness of $\mathcal{T}$ implies that of T (in the previous theorems, we have used the opposite direction: boundedness of T yields boundedness of $\mathcal{T}$. But it was not that immediate). This also applies to maximal operators with easy modifications. So immediate corollaries of our results on tent spaces are the followings.

Proposition A.3. *Let $\mathcal{M}$ be the centered Hardy-Littlewood maximal operator. We have, for all $1 < r < \infty$,*
 (a) $\mathcal{M} : (E^p_r)_t \to (E^p_r)_t$ *for all* $1 < p < \infty$.

 (b) $\mathcal{M} : (E^1_r)_t \to (wE^1_r)_t$.

Proposition A.4. *Let $\mathcal{T}$ be a Calderón-Zygmund operator of order $\delta \in (0,1]$. We have, for all $1 < r < \infty$,*
 (a) $\mathcal{T} : (E^p_r)_t \to (E^p_r)_t$, *for all* $1 < p < \infty$.

 (b) $\mathcal{T} : (E^1_r)_t \to (wE^1_r)_t$.

 (c) $\mathcal{T} : (\mathfrak{E}^p_r)_t \to (E^p_r)_t$, *for all* $\frac{n}{n+\delta} < p \leq 1$.

 (d) $\mathcal{T} : (\mathfrak{E}^p_r)_t \to (\mathfrak{E}^p_r)_t$, *for all* $\frac{n}{n+\delta} < p \leq 1$. *if* $\mathcal{T}^*(1) = 0$.

Here, $(\mathfrak{E}^p_r)_t$ is the space of functions in $(E^p_r)_t$ such that there exists an atomic decomposition $\sum_{i=1}^{\infty} \lambda_i a_i$ with $\int_{\mathbb{R}^n} a_i(x)\,dx = 0$, for all $i \in \mathbb{N}$. The atoms are defined in [14] for $r = 2$ and this adapts here. It suffices for understanding the statement to remark that $(\mathfrak{E}^p_r)_t = \pi(\mathfrak{T}^p_r)$.

Proposition A.5. *Let $\mathcal{M}_\alpha$ be the maximal fractional and $\mathcal{I}_\alpha$ the Riesz potential of order $\alpha \in (0,n)$. We have, for all $\frac{n}{n-\alpha} < r < \infty$ and $1 < p < q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$,*

$$\mathcal{M}_\alpha, \mathcal{I}_\alpha : (E^q_r)_t \to (E^p_r)_t.$$

Proposition A.6. *Let $\nabla L^{-\frac{1}{2}}$ be the Riesz transform associated to L. We have, for $q_-(L) < p, r < q_+(L)$,*

$$\nabla L^{-\frac{1}{2}} : (E^p_r)_t \to (E^p_r)_t.$$

Appendix B
INTERPOLATION

Let us defined the interpolation space described in [26] by A.P. Calderón.

Let A, B be Banach spaces embedded in a complex topological vector space V, and such that $\|\cdot\|_A$ and $\|\cdot\|_B$ denote the norm in A and B, respectively. Now, consider the space $A + B = \{x = y + z : y \in A,\ z \in B\}$ endowed with the norm

$$\|x\|_{A+B} := \inf\{\|y\|_A + \|z\|_B : x = y + z,\ y \in A,\ z \in B\}.$$

Then, the space $A + B$ becomes a Banach space.

Now, consider the linear space of functions $\mathcal{F} := \mathcal{F}(A, B)$ as the space of all functions $f(\xi)$, $\xi = \theta + it$, defined in the strip $0 \leq \theta \leq 1$ of the $\xi - plane$, with values in $A + B$ continues and bounded with respect to the norm of $A + B$ in $0 \leq \theta \leq 1$ and analytic in $0 < \theta < 1$, and such that $f(it) \in A$ is A-continuous and tends to zero as $|t|$ tends to infinity and $f(1 + it) \in B$ is B-continuous and tends to zero as $|t|$ tends to infinity. The norm that we consider in this space is the following

$$\|f\|_{\mathcal{F}} := \max\left\{\sup_t \|f(it)\|_A, \sup_t \|f(1 + it)\|_B\right\},$$

under this norm $\mathcal{F}$ becomes a Banach space.

Finally, for a given θ, $0 \leq \theta \leq 1$, we define the space $[A, B]_\theta := \{x \in A + B :\ x = f(\theta), f \in \mathcal{F}\}$ endowed with the norm

$$\|x\|_{[A,B]_\theta} := \inf\{\|f\|_{\mathcal{F}} : x = f(\theta)\}.$$

Then $[A, B]_\theta$ is a Banach space continuously embedded in $A + B$.

Bibliography

[1] A. Amenta, *Interpolation and embeddings of weighted tent spaces*, arXiv:1509.05699.

[2] A. Amenta,*Tent spaces over metric measure spaces under doubling and related assumptions*, Operator Theory: Advances and Applications, "Operator Theory in Harmonic and Non-commutative Analysis" (2014).

[3] P. Auscher, *On necessary and sufficient conditions for L^p estimates of Riesz transform associated to elliptic operators on $\mathbb{R}^n$ and related estimates*, Mem. Amer. Math. Soc., vol. 186, no. 871, March (2007).

[4] P. Auscher, *Change of Angles in tent spaces,* C. R. Math. Acad. Sci. Paris 349 (2011), no. 5-6, 297-301.

[5] P. Auscher, X.T. Duong, A. McIntosh, *Boundedness of Banach space valued singular integral operators and applications to Hardy spaces*, unpublished manuscript.

[6] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*, Ann. of Math. (2) 156 (2002), 633-654.

[7] P. Auscher, S. Hofmann, J.M. Martell, *Vertical versus conical square functions*. Trans. Amer. Math. Soc. 364 (2012), no. 10, 5469-5489.

[8] P. Auscher, C. Kriegler, S. Monniaux, P. Portal, *Singular integral operators on tent spaces*, J. Evol. Equ. 12 (2012), no. 4, 741-765.

[9] P. Auscher, J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights*, Advances in Mathematics 212 (2007), 225-276.

[10] P. Auscher, J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part II: off-diagonal estimates on spaces of homogeneous type*, J. Evol. Equ. 7 (2007), no. 2, 265-316.

[11] P. Auscher, J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part III: harmonic analysis of elliptic operators*, J. Funct. Anal. 241 (2006), 703-746.

[12] P. Auscher, A. McIntosh, E. Russ, *Hardy spaces of differential forms on Riemannian manifolds*, J. Geom. Anal. 18 (2008), no. 1, 192-248.

[13] P.Auscher, A. McIntosh, A. Morris, *Calderón Reproducing Formulas and Applications to Hardy Spaces*. Preprint (2013).

[14] P. Auscher, M. Mourgoglou, *Representation and uniqueness for boundary value elliptic problems via first order systems*, arXiv:1404.2687.

[15] P. Auscher, C. Prisuelos-Arribas, *Tent space boundedness via extrapolation*, Mathematische Zeitschrift (2016), DOI 10.1007/s00209-016-1814-7.

[16] P. Auscher, P. Tchamitchian, *Calcul fonctionnel précisé pour des opérateurs elliptiques complexes en dimension un (et applications à certaines équations elliptiques complexes en dimension deux),* Ann. Inst. Fourier , 45, 3, 721-778, (1995).

[17] P. Auscher, E. Russ, *Hardy spaces and divergence operators on strongly Lipschitz domain of* $\mathbb{R}^n$, J. Funct. Anal. 201 (2003), no. 1, 148-184.

[18] N. Bard *Real interpolation of Sobolev spaces*, Mathematica Scandinavica, vol. 31, no4, 235-264, (2009).

[19] J. Bergh, J. Löfström, *Interpolation spaces. An introduction*. Grundlehren der mathematischen Wissenschaften, no. 223. Springer-Verlag Berlin Heidelberg, New York, (1976).

[20] A. Bernal, *Some results on complex interpolation of* T_q^p *spaces*, Interpolation spaces and related topics (Ramat-Gan), Israel Mathematical Conference Proceedings , vol. 5, (1992), pp. 1-10.

[21] J.-P. Bertrandias, C. Datry, and C. Dupuis, *Unions et intersections d'espaces* L^p *invariantes par translation ou convolution*. Ann. Inst. Fourier (Grenoble), 28 (1978), pp. 53-84.

[22] T.A. Bui, J. Cao, L.D. Ky, D. Yang, S. Yang, *Musielak-Orlicz-Hardy Spaces Associated with Operators Satisfying Reinforced Off-Diagonal Estimates*, Analysis a Geometry in Metric Spaces, (2013), 69-129.

[23] T.A. Bui, J. Cao, L.D. Ky, D. Yang, S. Yang, *Weigthed Hardy spaces associated with opertors satisfying reinforced off-diagonal estimates*, Taiwanese Journal of Mathematics, vol. 17, no. 4, pp. 1127-1166, August (2013).

[24] T. A. Bui, X.T. Duong, *Weighted Hardy spaces associated to operators and boundedness of singular integrals*. Preprint (2012).

[25] R.C. Busby and H.A. Smith, *Product-convolution operators and mixed -norm spaces*. Trans. Amer. Math.Soc., 263 (1981), pp. 309-341.

[26] A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113-190.

[27] J. Cao, D-C. Chang, Z. Fu, D. Yang. *Real interpolation of weighted tent spaces*, Applicable Analysis. 95(11) (2015), 2415-2443.

[28] C. Carton-Lebrun, H. Heinig, S. Hofmann, *Integral operators on weighted amalgams* , Studia Math 109 (1994), 133-157.

[29] R.R. Coifman, *A real variable characterization of* H^p, Studia Math 51 (1974) 269-274.

[30] R.R. Coifman, C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Mathematica, T. Li. (1974).

[31] R.R. Coifman, Y. Meyer, *Ondelettes et opérateurs*. Hermann, volume 3, (1990).

[32] R.R. Coifman, Y. Meyer, E.M. Stein, *Some new function spaces and their applications to harmonic analysis*. J. Funct. Anal. 62(2), 304-335 (1985).

[33] R.R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Sot. 83 (1977), 569-645.

[34] M. Cowling, S. Meda, R. Pasquale, *Riesz potentials and amalgams*, Ann. Inst. Fourier, Grenoble 49, 4 (1999), 1345-1367.

[35] D.V. Cruz-Uribe, J.M. Martell, C. Pérez, *Weights extrapolation and the theory of Rubio de Francia*, Operator Theory: Advances and Applications, **215**. Birkhäuser/Springer Basel AG, Basel, (2011).

[36] G.P. Cubera, J. García-Cuerva, C. Pérez, *Extrapolation with weights, Rearrangement Invariant Function Spaces, Modular inequalities and applications to Singular Integrals,* Adv. Math. 203 (2006), no. 1, 256-318.

[37] J. Duoandikoetxea, *Fourier Analysis*, Grad. Stud. Math. 29, American Math. Soc., Providence, (2000).

[38] X.T. Duong, L.X. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. 18 (2005), no. 4, 943-973.

[39] X.T. Duong, L.X. Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. 58 (2005), no. 10, 1375-1420.

[40] C. Fefferman, E.M. Stein, *H^p spaces of several variables*. Acta Math. 129 (1972), no. 3-4, 137-193.

[41] C. Fefferman, E. M. Stein, *Some maximal inequalities*. Amer. J. Math. 93 (1971), 107-116.

[42] H.G. Feichtinger, *Banach convolution algebras of Wiener type*. Functions, Series, Operators, Proc. Conf. Budapest, 38, Colloq. Math. Soc. János Bolyai (1980), 509 524.

[43] H.G. Feichtinger, *Generalized amalgams, with applications to Fourier transform*. Canad. J. Math. 42(3), 395-409, (1990).

[44] J. Feuto, I. Fofana, K. Koua, *Weighted norm inequalities for a maximal operator in some subspace of amalgams*, arXiv:0901.4197.

[45] I. Fofana, *Continuité de l'intégral fractionnaire et espace $(L^p, \ell^q)^\alpha$*. Comptes Redus de Séances de l'Académie de Sciences I, vol. 308, pp. 525-527, (1989).

[46] J.J.F. Fournier, J. Stewart, *Amalgams of L^p and ℓ^q*. Bull. Amer. Math. Soc. (N.S.) 13 (1985), 1-21.

[47] J. García-Cuerva, *An extrapolation theorem in the theory of A_p weights*, Proc. Amer. Math. Soc. 87 (1983), no. 3, 422-426.

[48] J. García-Cuerva, *Weighted H^p spaces*, Dissertationes Math. (Rozprawy Mat.) 162 (1979).

[49] J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North Holland, Amsterdam, (1985).

[50] L. Grafakos, *Modern Fourier Analysis, second edition*. Graduate Texts in Mathematics, vol. 250, Springer, New York, (2009).

[51] E. Harboure, J. Torrea, and B. Viviani, *A vector-valued approach to tent spaces* , J. Anal. Math. 56 (1991), 125-140.

[52] C.E. Heil, *Wiener amalgam spaces in generalized harmonic analysis and wavelet theory*, Ph.D. Thesis, University of Maryland, College Park, MD, (1990). ftp://ftp.math.gatech.edu/pub/users/heil/thesis.pdf.

[53] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates.* Mem. Amer. Math. Soc., vol. 214, no. 1007 (third of 5 numbers), November (2011).

[54] S. Hofmann, J.M. Martell, *L^p bounds for Riesz Transforms and Square Roots Associated to Second Order Elliptic Operators*. Publ. Mat. 47 (2003), no. 2, 497-515.

[55] S. Hofmann, S. Mayboroda, *Hardy and BMO spaces to divergence form elliptic operators.* Math. Ann. 344 (2009), no. 1, 37-116.

[56] S. Hofmann, S. Mayboroda, A. McIntosh, *Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces.* Ann. Sci. École. Norm. Sup., 44 (2011) pp. 723-800.

[57] F. Holland, *Harmonic analysis on amalgams of L^p and ℓ^q,* J. London Math. Soc., 10 (1975), pp. 295-305.

[58] R. Hunt, B. Muckenhoupt, R.Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*. Trans. Amer. Math. Soc. 176 (1973), 227-251.

[59] T. Hytönen, J. van Neerven, P. Portal, *Conical square function estimates in UMD Banach spaces and applications to $H^\infty-$functional calculi.* J. Anal. Math. 106 (2008), 317-351.

[60] T. Hytönen, L. Weis, *The Banach Space-valued BMO, Carleson's Condition, and Paraproducts.* J. Fourier Anal. Appl. 16 (2010), 495-513.

[61] R. Jiang, D. Yang, *New Orlicz-Hardy spaces associated with divergence form elliptic operators*, J. Funct. Anal. 258 (2010), no. 4, 1167-1224.

[62] T. Kato, *Perturbation theory for linear operators*. Springer Verlag, New York, (1966).

[63] N. Kikuchi, E. Nakai, N. Tomita, K. Yabuta, T. Yoneda, *Calderón-Zygmund operators on amalgam spaces and in the discrete case*. J. Math. Anal. Appl. 335 (2007) 198-212.

[64] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations.* Adv. Math., 157(1): 22-35, (2001).

[65] R.H. Latter *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math. 62 (1978), 93-101.

[66] A.K. Lerner, *On sharp aperture-weighted estimates for square functions*. Preprint (2013).

[67] S. Liu, L. Song, *An atomic decomposition of weighted Hardy spaces associated to self-adjoint operators*, J. Funct. Anal. 265 (2013), no. 11, 2709-2723.

[68] J.M. Martell, C. Prisuelos-Arribas, *Weighted Hardy spaces associated with elliptic operators. Part I: weighted norm inequalities for conical square functions*, to appear in Trans. Amer. Soc., arXiv:1406.6285.

[69] J.M. Martell, C. Prisuelos-Arribas, *Weighted Hardy spaces associated with elliptic operators. Part II: Characterizations of $H^1_L(w)$* , to appear in Publ. Mat., arXiv:1701.00920.

[70] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*. Trans. Amer. Soc. 165 (1972), pp. 207-226.

[71] B. Muckenhoupt, R.L. Wheeden, *Weighted norm inequalities for fractional integrals*. Trans. Amer. Math. Soc. vol. 192, (1974).

[72] C. Prisuelos-Arribas *Weighted Hardy spaces associated with elliptic operators. Part III: Characterizations of $H^p_L(w)$ and the weighted Hardy space associated with the Riesz transform*. In preparation.

[73] J.L. Rubio de Francia, *Factorization theory and A_p weights*, Amer. J. Math. 106 (1984), no. 3, 533-547.

[74] E. Russ, *The atomic decomposition for tent spaces on spaces of homogeneous type.* Proc. Center Math. Appl., Australian National University, 42. Australian National University, Canberra, (2007).

[75] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series. no. 30, Princeton University Press, Princeton, N.J. (1970).

[76] E.M. Stein, G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of H^p-spaces*, Acta Math. 103 (1960), 25-62.

[77] J. Stewart, *Fourier transforms of unbounded measures*, Canad. J. Math., 31 (1979), pp. 1281-1292.

[78] J-O. Strömberg, A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Mathematics, 1381. Springer-Verlag, Berlin, (1989).

[79] A. Torchinsky, *Real-variable methods in harmonic analysis*. Academic Press, INC, (1986).

[80] N. Wiener, *On the representation of functions by trigonometric integrals*. Math. Z. 24, (1926), 575-616.

[81] N. Wiener, *Tauberian theorems,* Ann. of Math. 33 (1932), 1-100.